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PVA EXPO PRAGUE

Solutions

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Solutions of problems

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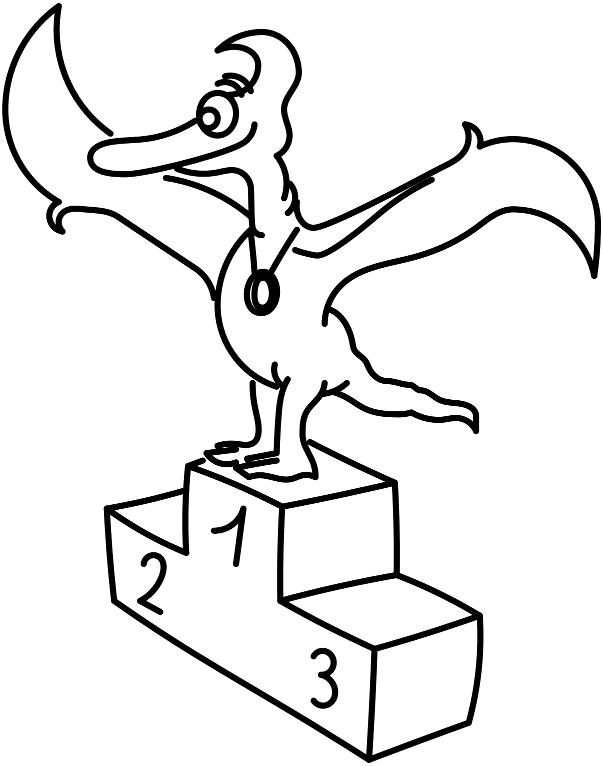

Problem AA ... jellyfSIh

During one of his explorations, the space traveler Vojta stumbled upon a planet inhabited by a unique species of intelligent jellyfish. Since you hardly ever see anything like that, he immediately began to explore this world. He learned, for example, that the inhabitants were very skilled in physics. After observing them for a while, he discovered that their equivalent of the SI system had, among others, the quantities voltage, charge, and frequency. He was also able to determine the following conversion relationships.

$$\textit{voltage: } 1\,\wr \doteq 541\,\mathrm{V}\,, \quad \textit{charge: } 1\,\circledast \doteq 0.301\,\mathrm{C}\,, \quad \textit{frequency: } 1\,\smile \doteq 2.93\,\mathrm{Hz}\,.$$

Before taking off, a resident asked him about his rocket. Vojta, eager to impress, excitedly planned to tell him that his rocket had 800 GW of power – but realized that the jellyfish wouldn't understand. Express the power of Vojta's rocket in units used by jellyfish.

Volta would like to live like a jellyfish.

We aim to express the unit of power, the watt, in the units available to us – in volts, coulombs, and hertz. Once we know this expression, we can proceed much as we would, for example, to convert meters per second to miles per hour, which will be enough to plug in the conversion relations.

To find that expression of the watt, we need to compare the *dimensions* of the units available to us. We'll express everything first in the base SI units.

$$1\,\mathrm{V} = 1\,\mathrm{kg{\cdot}m^2{\cdot}s^{-3}{\cdot}A^{-1}}\,, \quad 1\,\mathrm{C} = 1\,\mathrm{A{\cdot}s}\,, \quad 1\,\mathrm{Hz} = 1\,\mathrm{s^{-1}}\,, \quad 1\,\mathrm{W} = 1\,\mathrm{kg{\cdot}m^2{\cdot}s^{-3}}\,.$$

From these breakdowns, it is not difficult to see that $1\,\mathrm{W} = 1\,\mathrm{V} \cdot 1\,\mathrm{C} \cdot 1\,\mathrm{Hz}$. If we add the extraterrestrial units, we get

$$1\,\mathrm{W} = \frac{1}{541}\,\wr \cdot \frac{1}{0.301}\,\circledast \cdot \frac{1}{2.93}\,\smile \doteq 0.00210\,\wr \cdot \circledast \cdot \smile\,.$$

We can easily get the result $800\,\mathrm{GW} \doteq 1.68 \cdot 10^9\,\wr \cdot \circledast \cdot \smile$.

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Problem AB ... bathing in a bathtub

Danka is filling her bathtub. From the tap, $Q = 3.00\,\mathrm{dl{\cdot}s^{-1}}$ of water flows into it. For how long, at most, can Danka fill the bathtub to the point where she can fully immerse herself without the water spilling over the tub's edge? The bathtub has an elliptical bottom with a major semi-axis of length $a = 70\,\mathrm{cm}$ and a minor semi-axis of length $b = 35\,\mathrm{cm}$. The height of the walls of the tub is $h = 50\,\mathrm{cm}$ and the walls are perpendicular to the bottom. Danka weighs $m = 55\,\mathrm{kg}$. Consider the density of the human body after inhalation $\rho_{\mathrm{D}} = 945\,\mathrm{kg{\cdot}m^{-3}}$. Give the result in the form mm:ss.

Danka was filling her bathtub.

The volume of the bathtub is calculated as the product of the area of its base and the height of the bathtub. Since the base is an ellipse, we calculate its area using the formula $S = \pi ab$. Then the volume of the bath is

$$V = \pi abh\,.$$

The volume of Danka's body can be calculated simply as

$$V_\mathrm{D} = \frac{m}{\rho_\mathrm{D}}\,.$$

The volume of water that can be filled into the bath is then given by the difference between the volumes of the bath and Danka's body, which is

$$V_\mathrm{v} = V - V_\mathrm{D} = \pi abh - \frac{m}{\rho_\mathrm{D}}\,.$$

At the same time, $V_\mathrm{v} = Qt$, where Q is the volumetric flow rate of water into the bath and t is the filling time we are searching for. Putting the last two relations as equal, we only need to express the time and plug in the numerical values

$$t = \frac{V - V_\mathrm{D}}{Q} = \frac{\pi abh - \frac{m}{\rho_\mathrm{D}}}{Q}\,,$$
$$t \doteq 1090\,\mathrm{s} = 18\,\mathrm{min}\,10\,\mathrm{s}\,.$$

Danka can fill the tub for a maximum of 18 minutes and 10 seconds.

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Problem AC ... panic at the escalators

You might find yourself ascending an escalator when you suddenly realize the need to backtrack. The escalators travel at a speed $u = 0.65\,\mathrm{m{\cdot}s^{-1}}$ and you are able to maintain a running speed $v = 6.0\,\mathrm{km{\cdot}h^{-1}}$. In what greatest distance from the bottom of the stairs is it still worth turning around and running down the stairs versus running up and then switching to the escalator going down and running down it? Write the result as a ratio to the total length of the escalators.

Disregard the time spent transitioning to the other escalator at the top and suppose that you run upwards and downwards equally fast (ascending is more tiring while descending demands increased attentiveness). Additionally, assume the absence of any obstacles on the stairs.

Karel wondered what to do.

Let x be the required distance from the start of the stairs and l the length of the escalator. Next, we will denote the time it will take a person to turn around and run down against the motion of the stairs by t_1. In this scenario, the person will be moving relative to the ground at a velocity of $v - u$

$$t_1 = \frac{x}{v-u}\,.$$

Then, let t_2 represent the time it will take him to ascend the remaining part of the escalator on which he is currently standing at a speed of $v + u$ relative to the ground and then descend again on the other escalator at a speed of $v + u$. Therefore

$$t_2 = \frac{l-x}{v+u} + \frac{l}{v+u} = \frac{2l-x}{v+u}\,.$$

In the point x, the times t_1 and t_2 are the same. Thus, it is sufficient to put the expressions for both times into the equation and express the ratio x/l

$$\begin{aligned}\frac{x}{v-u} &= \frac{2l-x}{v+u}\,,\\ x(v+u) &= (2l-x)(v-u)\,,\\ x(v+u)+x(v-u) &= 2l(v-u)\,,\\ 2xv &= 2l(v-u)\,,\\ \frac{x}{l} &= \frac{v-u}{v} = 1-\frac{u}{v}\,,\\ \frac{x}{l} &= 0.61\,.\end{aligned}$$

We can see that it is worth turning around and coming back if we are at most at a distance from the top of the stairs equal to 0.61 the length of the escalator.

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Problem AD ... storing heat in liquids

We would like to store energy in a liquid by heating it by a fixed temperature difference ΔT and then store it in a thermally insulated container for a fixed time. The container has a limited volume and we would like to store as much energy in it as possible. Which is better suited for energy storage, water, or mercury? How many times better? For the result, give the ratio of the heat stored in water to the heat stored in mercury. *Karel was thinking about energy storage.*

Let us denote the volume of the container V. For each of the liquids, we can determine its mass in the container from its density ρ as $m=\rho V$. The heat stored by the liquid in the container is then obtained from the calorimetric equation as

$$Q = mc\Delta T = \rho c V\Delta T\,,$$

where c is the specific heat capacity of the liquid. The ratio of the heat stored in water to the heat stored in mercury is given as

$$w = \frac{Q_{\mathrm{water}}}{Q_{\mathrm{Hg}}} = \frac{\rho_{\mathrm{water}}c_{\mathrm{water}}}{\rho_{\mathrm{Hg}}c_{\mathrm{Hg}}}\,.$$

Using the values $\rho_{\mathrm{water}} = 0.998\,\mathrm{g\cdot ml^{-1}}$, $\rho_{\mathrm{Hg}} = 13.5\,\mathrm{g\cdot ml^{-1}}$, $c_{\mathrm{water}} = 4.184\,\mathrm{J\cdot g^{-1}\cdot K^{-1}}$, $c_{\mathrm{Hg}} = 0.14\,\mathrm{J\cdot g^{-1}\cdot K^{-1}}$ we get a value for the ratio $w \doteq 2.21$. Thus, we see that by using water we store about twice as much energy in the container compared to mercury.

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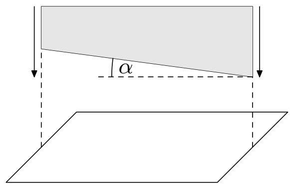
Problem AE ... sheet shears

A sharp, heavy, straight blade is dropped from above (like a guillotine) from $h = 8\,\mathrm{m}$ on very thin paper. The paper lies horizontally, the blade falls, and its underside is inclined to the horizontal plane by an angle $\alpha = 4 \cdot 10^{-5}\,\mathrm{mrad}$. What is the speed of the point in which the paper is cut? Neglect the drag forces and the deceleration of the blade due to interaction with the paper.



Jarda used to build paper cutouts.

From the conservation of mechanical energy, the vertical speed of the blade is

$$mgh = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{2gh} = 12.53\,\mathrm{m{\cdot}s^{-1}}\,.$$

The horizontal speed of the point where the cutting edge meets the paper is (clearly visible in the figure from the problem statement)

$$u = \frac{v}{\tan\alpha} = 3.13 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}\,.$$

Therefore, the paper is being cut faster than the speed of light. However, the result is not physically meaningless because no information propagates at this speed. Moreover, the speed would be infinite if the blade were perfectly aligned with the horizontal plane!

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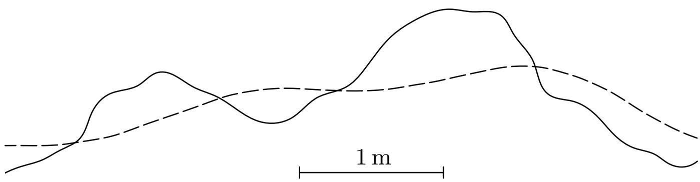
Problem AF ... looking for a cyclist

During a FYKOS party, Viktor decided to get some fresh air and went for a bike ride. However, he did not mention that to anyone, and after a while, a group of other organizers decided to look for him. They immediately found bike tracks in the snow and it was obvious to them that in such weather, nobody except Viktor would ride a bike. To continue the search, they needed to find out in which direction he went and also the distance between the centers of his bike's (equally large) wheels. Determine these parameters from the provided picture of the tracks.



Viktor's bicycle is still taking up space in the hallway.

Let's assume that a bike has two identical wheels, the front wheel can freely turn, while the rear wheel is fixed in the bike frame. Due to this, the plane of the rear wheel contains the center of the front wheel. The direction of the rear wheel must always be towards the front wheel. This logic also holds for points of contact between the wheels and the ground, not just their centers. Therefore, we can conclude that the tangent to the track of the rear wheel always points towards the point on the front wheel's path, and especially that the distance between these points is constant and equal to the distance between the wheels.

For the moment, let's assume that the dashed line corresponds to the front wheel, and focus on the top right curve of the full line. When this curve is created by the rear wheel, for the tangent to point towards the other track, the cyclist would have to ride left to right on the right side of the curve. Similarly, however, he would need to ride right to left on the left side of the curve. It is clear that the rear wheel corresponds to the dashed line and the front wheel to the full line.

If the cyclist was moving left to right, then immediately on the left side, the tangent to the dashed line would give us a ray which intersects the full line only at the other side of the picture (if at all). Elsewhere, we would get much smaller distances, and also considering the scale, we can conclude that this option cannot be correct. The cyclist must have been going right to left.

To measure the distances, we start by placing the ruler on the right side. We find out that the distance between the wheels is a bit greater than one meter. This lets us exclude incorrect data, which could happen if a tangent crosses the full line multiple times, from the next measurements.

From a single measurement, however, due to limited size of the picture and inaccuracy of determining the tangent, we might be unable to find the correct solution. In order to increase the probability of success, we should repeat the measurement multiple times in different places.

We have measured values 3.35 cm, 3.20 cm, 3.25 cm, 3.25 cm, 3.30 cm, 3.40 cm, 3.60 cm and the length of the scale 2.95 cm. The average of these values is (3.34 ± 0.05) cm, which corresponds to the distance between the wheels $(3.34\pm0.05)/2.95\,\mathrm{m} = (113\pm2)\,\mathrm{cm}$. The correct result should be 110 cm, so we were quite successful.

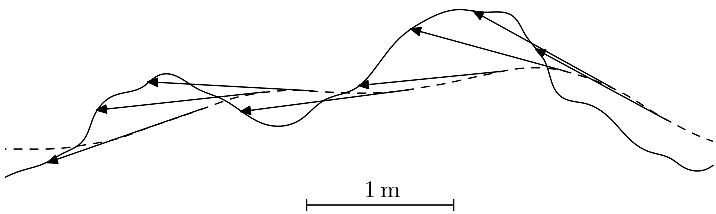


Fig. 1: Example of a possible solution.

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Problem AG ... clean energy of tomorrow

The controlled fission reaction is a process in which neutrons merge with the uranium-235 nuclei, which then fission into lighter elements, releasing more neutrons in the process. But there's a catch – the reaction works best at low neutron energies ($0.025\,\mathrm{eV}$), while the neutrons released in a fission reaction are fast (energies on the order of MeV). Therefore, a material to slow down the neutrons, called a moderator, is used in nuclear reactors. It must meet special requirements – in particular, it must have light nuclei so that neutrons lose as much energy as possible in elastic and inelastic collisions, and it must not absorb neutrons. The Czech *nuclear power plants Temelín and Dukovany use water as a moderator, in which neutrons are slowed down by collisions with hydrogen nuclei. Assume that a fast neutron undergoes an elastic collision with a stationary hydrogen nucleus. What is the largest percentage of its kinetic energy that the fast neutron can lose in the collision, when considering the same mass of proton and neutron?*

This problem is brought to you by CEZ Group.

Jindra was pondering about losses of energy.

Substances used as moderators are, for example, water (hydrogen), heavy water (deuterium/heavy hydrogen) or graphite (carbon). The nuclear reactors at Temelín and Dukovany are pressurised water reactors. This type of reactor uses ordinary water as a moderator. The advantage of water as a moderator is that it can simultaneously be used to cool down the reactor and to heat up the water in the secondary cooling circuit. The steam in the secondary circuit is then used to rotate the turbines to generate electricity. The Dalešice reservoir serves as a source of cooling water for the Dukovany power plant and the Hněvkovice reservoir serves as a source of cooling water for the Temelín power plant. It should be stressed here that the water from the water tanks is not used directly in the reactor as a moderator, but is used to cool the steam in the secondary cooling circuit. Most of the hydrogen in water molecules is light hydrogen ${}^1_1\mathrm{H}$ with one proton in the nucleus. The isotope ${}^2_1\mathrm{H}$ (deuterium) with one proton and one neutron in the nucleus is only approximately 1 atom out of 6 400.

In particle physics, the unit of energy commonly used is the electron volt $1\,\mathrm{eV} = 1.602{\cdot}10^{-19}\,\mathrm{J}$. Multiples of electron volts are also used, kiloelectron volt $1\,\mathrm{keV} = 1\,000\,\mathrm{eV}$, megaelectron volt $1\,\mathrm{MeV} = 1\,000\,000\,\mathrm{eV}$, and others. The rest mass of the proton is $m_\mathrm{p} = 1.672\,6 \cdot 10^{-27}\,\mathrm{kg} = 938.27\,\mathrm{MeV{\cdot}c^{-2}}$. The rest mass of the neutron is slightly higher $m_\mathrm{n} = 1.674\,9 \cdot 10^{-27}\,\mathrm{kg} = 939.57\,\mathrm{MeV{\cdot}c^{-2}}$, but for this problem we will assume that the proton and neutron have the same mass. We have expressed this in units of megaelectron volts per speed of light squared, which you can verify has a dimension of mass, and $1\,\mathrm{MeV{\cdot}c^{-2}} = 1.783 \cdot 10^{-30}\,\mathrm{kg}$.

First of all, we need to clarify the question whether it is correct to calculate the collision of a neutron with a proton in a hydrogen nucleus and whether the rest of the water molecule will interfere with the collision process. The binding energy of the H−O bond in the water molecule is 5.5 eV. The binding energy of the electron to the hydrogen nucleus is 13.6 eV. If the neutron arrives with a much higher kinetic energy than these two values, then we can think of the hydrogen nucleus (proton) as a free particle that is not affected by the rest of the water molecule. Since, according to the information in the problem statement, a neutron is involved in the collision (kinetic energy on the order of MeV), it is OK to think of the neutron as colliding with a free proton.

Next, we must verify that the special theory of relativity does not apply here (it is neglectable). The relation for relativistic kinetic energy is

$$E_\mathrm{k} = (\gamma - 1)mc^2\,, \tag{1}$$

where γ is the Lorentz factor, m is the rest mass of the object and c is the speed of light. The Lorentz factor depends on the velocity of the object v according to the relation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,.$$

If the $\gamma - 1$ term in the relation (1) is much smaller than 1, then the particle is not relativistic and thus we can use the relation for kinetic energy from classical physics

$$E_\mathrm{k} = \frac{1}{2}mv^2 \,.$$

The average neutron released in the fission of a uranium-235 nucleus has kinetic energy $E_{\mathrm{n,k}} \approx \approx 2\,\mathrm{MeV}$. For it holds

$$\gamma - 1 = \frac{E_{\mathrm{n,k}}}{m_\mathrm{n} c^2} \doteq 2.12 \cdot 10^{-3} \ll 1 \,.$$

Hence it can be considered as a non-relativistic particle.

In the elastic collision of a neutron with a proton, kinetic energy and momentum are conserved. As mentioned in the problem statement, we will consider that the masses of the neutron and the proton are equal to $m_\mathrm{n} \approx m_\mathrm{p} = m_0$. The initial kinetic energy $T_{\mathrm{n},1}$ and momentum $\mathbf{p}_{\mathrm{n},1}$ of the neutron are

$$T_{\mathrm{n},1} = \frac{1}{2} m_0 v_{\mathrm{n},1}^2 , \quad \mathbf{p}_{\mathrm{n},1} = m_0 \mathbf{v}_{\mathrm{n},1} \,,$$

where $\mathbf{v}_{\mathrm{n},1} = (v_{\mathrm{n},1}; 0; 0)$ is the velocity of the neutron as it approaches the stationary proton. For simplicity, we have chosen the positive direction of the x-axis to be parallel to the neutron velocity vector. The initial kinetic energy $T_{\mathrm{p},1} = 0$ and momentum $\mathbf{p}_{\mathrm{p},1} = \mathbf{0}$ are zero.

After the collision, the neutron flies away with the velocity $\mathbf{v}_{\mathrm{n},2}$ and the proton with the velocity $\mathbf{v}_{\mathrm{p},2}$. These velocities must satisfy the following system of equations

$$\frac{1}{2} m_0 v_{\mathrm{n},1}^2 = \frac{1}{2} m_0 v_{\mathrm{n},2}^2 + \frac{1}{2} m_0 v_{\mathrm{p},2}^2 \,,$$
$$m_0 \mathbf{v}_{\mathrm{n},1} = m_0 \mathbf{v}_{\mathrm{n},2} + m_0 \mathbf{v}_{\mathrm{p},2} \,.$$

This system has infinitely many solutions $\mathbf{v}_{\mathrm{n},2}$ and $\mathbf{v}_{\mathrm{p},2}$, but we are only interested in the solution where the neutron flies away with the lowest kinetic energy. And finding that solution is not difficult. By assuming equal masses for the proton and neutron, we can truncate m_0 in both equations, so we are just comparing the velocities. We can see the trivial solution $\mathbf{v}_{\mathrm{n},2} = \mathbf{0}$ and $\mathbf{v}_{\mathrm{p},2} = \mathbf{v}_{\mathrm{n},1}$, in which the neutron loses all its kinetic energy. Since kinetic energy cannot become negative, this answers the question of what is the largest percentage of kinetic energy a neutron can lose. Thus, in an ideal collision geometry, the neutron will lose all of its kinetic energy, i.e. $100\,\%$.

Appendix: General solution of elastic collision of two solid objects

In this problem, we found it very helpful to consider the masses of the proton and the neutron to be equal. But how should we solve the problem if we took into account the different masses of the particles? What if the proton had a non-zero initial velocity?

Consider a particle A with mass m_A, and initial velocity $\mathbf{v}_{\mathrm{A},1}$ and a particle B with mass m_B, and initial velocity $\mathbf{v}_{\mathrm{B},1}$. These particles collide and fly away with velocities $\mathbf{v}_{\mathrm{A},2}$, and $\mathbf{v}_{\mathrm{B},2}$. In an elastic collision, kinetic energy and momentum are conserved

$$\frac{1}{2} m_\mathrm{A} v_{\mathrm{A},1}^2 + \frac{1}{2} m_\mathrm{B} v_{\mathrm{B},1}^2 = \frac{1}{2} m_\mathrm{A} v_{\mathrm{A},2}^2 + \frac{1}{2} m_\mathrm{B} v_{\mathrm{B},2}^2 \,,$$
$$m_\mathrm{A} \mathbf{v}_{\mathrm{A},1} + m_\mathrm{B} \mathbf{v}_{\mathrm{B},1} = m_\mathrm{A} \mathbf{v}_{\mathrm{A},2} + m_\mathrm{B} \mathbf{v}_{\mathrm{B},2} \,.$$

This equation has infinitely many solutions for the velocity vectors $\mathbf{v}_{\mathrm{A},2}$, $\mathbf{v}_{\mathrm{B},2}$ and it is not simple to identify at least one solution, let alone the one with the lowest kinetic energy of the departing particle A. The problem is greatly simplified if we go to the reference frame associated with the center of gravity of both solid objects. The centre of gravity in the laboratory system moves at the speed of

$$\mathbf{v}_{\mathrm{t}} = \frac{m_{\mathrm{A}}\mathbf{v}_{\mathrm{A},1} + m_{\mathrm{B}}\mathbf{v}_{\mathrm{B},1}}{m_{\mathrm{A}} + m_{\mathrm{B}}} \,.$$

We switch to the centre of gravity reference frame (the velocities in it will be denoted by $\mathbf{u}$) by subtracting the centre of gravity velocity from the velocities $\mathbf{v}$ relative to the laboratory frame

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_{\mathrm{t}} \,. \tag{2}$$

In the centre of gravity reference frame, the total momentum of both particles is equal to zero

$$\mathbf{p}_{\mathrm{t}} = m_{\mathrm{A}}\mathbf{u}_{\mathrm{A},1} + m_{\mathrm{B}}\mathbf{u}_{\mathrm{B},1} = \mathbf{0} \,,$$

which can be verified by substituting for $\mathbf{u}$ from the relation (2). The vectors of the initial particle velocities $\mathbf{u}_{\mathrm{A},1}$, $\mathbf{u}_{\mathrm{B},1}$ lie on the same line and point in opposite directions. Similarly, the vectors of the final velocities $\mathbf{u}_{\mathrm{A},2}$, $\mathbf{u}_{\mathrm{B},2}$ lie on the same line and point away from each other, although they are not necessarily on the same line as the initial velocities. The two lines define a plane, so the collision of two particles in a center of gravity frame is a phenomenon taking place in a 2D plane.

Even in a centre of gravity reference frame, the collision is governed by the laws of conservation of energy and momentum

$$\frac{1}{2}m_{\mathrm{A}}u_{\mathrm{A},1}^2 + \frac{1}{2}m_{\mathrm{B}}u_{\mathrm{B},1}^2 = \frac{1}{2}m_{\mathrm{A}}u_{\mathrm{A},2}^2 + \frac{1}{2}m_{\mathrm{B}}u_{\mathrm{B},2}^2 \,,$$
$$m_{\mathrm{A}}\mathbf{u}_{\mathrm{A},1} + m_{\mathrm{B}}\mathbf{u}_{\mathrm{B},1} = m_{\mathrm{A}}\mathbf{u}_{\mathrm{A},2} + m_{\mathrm{B}}\mathbf{u}_{\mathrm{B},2} = \mathbf{0} \,.$$

The only possible solution is that the magnitudes of the initial and final velocities are equal

$$|\mathbf{u}_{\mathrm{A},1}| = |\mathbf{u}_{\mathrm{A},2}| = u_{\mathrm{A},1} = u_{\mathrm{A},2} \,, \quad |\mathbf{u}_{\mathrm{B},1}| = |\mathbf{u}_{\mathrm{B},2}| = u_{\mathrm{B},1} = u_{\mathrm{B},2} \,.$$

The direction of the final velocity vector is a free parameter of the collision. To determine it, we would need to find additional information about what the impulse vector was during the collision. However, it is easier to find a solution in the center of gravity frame where the particle A flies away with the lowest possible kinetic energy in the laboratory frame.

We choose the coordinate system so that the x-axis lies parallel to the direction of the initial particle velocities and the y-axis lies in the plane of the collision. The initial and final particle velocities are

$$\begin{aligned}
\mathbf{u}_{\mathrm{A},1} &= u_A(1;0;0) \,, \\
\mathbf{u}_{\mathrm{B},1} &= u_B(-1;0;0) \,, \\
\mathbf{u}_{\mathrm{A},2} &= u_A(\cos\theta;\sin\theta;0) \,, \\
\mathbf{u}_{\mathrm{B},2} &= u_B(-\cos\theta;-\sin\theta;0) \,,
\end{aligned}$$

where θ is the angle between the end and initial velocity vectors of particle A.

In the laboratory reference frame, particle A will fly away with the velocity

$$\mathbf{v}_{\mathrm{A},2} = \mathbf{u}_{\mathrm{A},2} + \mathbf{v}_{\mathrm{t}} \,.$$

The kinetic energy of the particle A will be the lowest when the square of the velocity

$$|\mathbf{v}_{\mathrm{A},2}|^2 = (u_A \cos\theta + v_{\mathrm{t},x})^2 + (u_A \sin\theta + v_{\mathrm{t},y})^2$$

will be the lowest. In this case, the components of the centre of gravity velocity $v_{\mathrm{t},x}$, $v_{\mathrm{t},y}$ must be expressed relative to the axes of the centre of gravity coordinate system. In other words, the origin of the center of gravity coordinate system moves at the velocity $\mathbf{v}_\mathrm{t}$.

In the case of a neutron with a mass of m_n striking a stationary proton with a mass of m_p, the direction of the x axis in the center of gravity frame is the same as the direction of the x axis in the laboratory frame, so the velocity of the center of gravity frame relative to the laboratory frame is

$$\mathbf{v}_\mathrm{t} = \frac{m_\mathrm{n}\mathbf{v}_{\mathrm{n},1}}{m_\mathrm{n}+m_\mathrm{p}} = \frac{m_\mathrm{n}v_{\mathrm{n},1}}{m_\mathrm{n}+m_\mathrm{p}}(1;0;0)\,.$$

The velocity of a neutron in a center of gravity reference frame is

$$\mathbf{u}_n = \mathbf{v}_{\mathrm{n},1} - \mathbf{v}_\mathrm{t} = v_{\mathrm{n},1}(1;0;0) - \frac{m_\mathrm{n}v_{\mathrm{n},1}}{m_\mathrm{n}+m_\mathrm{p}}(1;0;0) = \frac{m_\mathrm{p}v_{\mathrm{n},1}}{m_\mathrm{n}+m_\mathrm{p}}(1;0;0)\,.$$

The square of the final velocity of the neutron in the laboratory system

$$|\mathbf{v}_{\mathrm{n},2}|^2 = (u_n\cos\theta + v_{\mathrm{t},x})^2 + (u_n\sin\theta + v_{\mathrm{t},y})^2 = \left(\frac{m_\mathrm{p}v_{\mathrm{n},1}\cos\theta}{m_\mathrm{n}+m_\mathrm{p}} + \frac{m_\mathrm{n}v_{\mathrm{n},1}}{m_\mathrm{n}+m_\mathrm{p}}\right)^2 + \left(\frac{m_\mathrm{p}v_{\mathrm{n},1}\sin\theta}{m_\mathrm{n}+m_\mathrm{p}}\right)^2$$

is the lowest for $\theta = 180^\circ$. Thus, the neutron loses the most kinetic energy in a head-on collision when it rotates direction by 180° in the center of gravity frame. Accounting for the different masses of the neutron and the proton, the neutron would lose at most

$$\eta = 1 - \left(\frac{m_\mathrm{n}-m_\mathrm{p}}{m_\mathrm{n}+m_\mathrm{p}}\right)^2 \doteq 99.999953\,\%$$

of its kinetic energy.

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Problem AH ... speed up at crossroads

During his last car trip, Karel was wondering how much time he would save if he was accelerating (and also decelerating) twice as much as he did. Karel was driving in the following uniform ways during the following segments of the trip:

- $t_0 = 9.0\,\mathrm{min}$ *of standing still,*
- $t_1 = 8.0\,\mathrm{min}$ *of uniform acceleration from* $v_0 = 0\,\mathrm{km{\cdot}h^{-1}}$ *to* $v_{50} = 50.0\,\mathrm{km{\cdot}h^{-1}}$,
- $t_2 = 8.0\,\mathrm{min}$ *of uniform deceleration from* v_{50} *to* v_0,
- $t_3 = 12.0\,\mathrm{min}$ *of driving with uniform speed* v_{50},
- $t_4 = 4.0\,\mathrm{min}$ *of uniform acceleration from* v_{50} *to* $v_{90} = 90.0\,\mathrm{km{\cdot}h^{-1}}$,
- $t_5 = 4.0\,\mathrm{min}$ *of uniform deceleration from* v_{90} *to* v_{50} *a*
- $t_6 = 15.0\,\mathrm{min}$ *of driving with uniform speed* v_{90}.

Do not forget that Karel has to drive along the same path from start to end (so the same speed limits apply). Also assume that all other driving conditions are the same (e.g. the time of waiting at crossroads does not change and the road is otherwise empty) and Karel is trying to drive as fast as possible within speed limits in both cases.

Karel was wondering if it is worth doing.

The time of the original trip can be found easily as

$$T = t_0 + t_1 + t_2 + t_3 + t_4 + t_5 + t_6 = 60\,\mathrm{min}\,.$$

We will denote the times during the second trip, with double acceleration, by the prime symbol. The first time $t_0' = 9\,\mathrm{min}$ is directly given in the problem statement, since that obviously does not change.

Next, we should realize that when moving with uniform acceleration, the traversed distance is generally

$$s = v_0 t + \frac{1}{2} a t^2\,,$$

where v_0 is the initial speed, a is the acceleration and t is time. If we are given the minimum and maximum speed between which we are accelerating and we need to spend the same time on this motion, the traversed distance is always the same, whether we are accelerating once or multiple times. This can be shown using the additional formula

$$v = v_0 + at\,,$$

where v is the final speed after time t. Then we see that if v and v_0 are constant and we split the time t into N identical time segments, the resulting acceleration needs to be Na. The traversed distance is the sum of distances

$$s = \sum_{i=1}^{N} \left(v_0 \frac{t}{N} + \frac{1}{2} N a \left(\frac{t}{N}\right)^2 \right) = N \left(\frac{1}{N} v_0 t + \frac{1}{N}\frac{1}{2} a t^2 \right) = v_0 t + \frac{1}{2} a t^2\,,$$

which is the same distance. This is why we could simply put total times spent on the segments of the trip into the problem statement without specifying in how many spots acceleration was involved. On the other hand, if we are not limited by the time spent on accelerating, but want to accelerate on some path and can accelerate twice as fast, we reach the goal somewhat faster.

We need to know the lengths of segments at which speed is limited to $v_{50} = 50\,\mathrm{km{\cdot}h^{-1}}$ and where it is limited to $v_{90} = 90\,\mathrm{km{\cdot}h^{-1}}$. From the logic of the problem statement, we may assume that the speed limit of $50\,\mathrm{km{\cdot}h^{-1}}$ is at the segments traversed during times t_1, t_2 and t_3, while the limit of $90\,\mathrm{km{\cdot}h^{-1}}$ is at the segments traversed during t_4, t_5 and t_5.

The distances traversed during times t_1 and t_2 are identical (it does not matter if we are accelerating from the first speed to the second one or decelerating from the second speed to the first one, when the times of accelerating and decelerating are the same). The initial speed is zero, so the formula simplifies to

$$s_1 = s_2 = \frac{1}{2} \frac{v_{50}}{t_1} t_1^2 = \frac{1}{2} v_{50} t_1 \doteq 3.3\,\mathrm{km}\,.$$

The distance traversed during the time t_3 is obtained as

$$s_3 = v_{50} t_3 = 10\,\mathrm{km}\,.$$

Similarly, the distances traversed during times t_4 and t_5 are the same

$$s_4 = s_5 = v_{50}t_4 + \frac{1}{2}\frac{v_{90} - v_{50}}{t_4}t_4^2 = \frac{1}{2}\left(v_{90} + v_{50}\right)t_4 \doteq 4.7\,\mathrm{km}\,.$$

Again, we simply obtain the last segment of the trip

$$s_6 = v_{90}t_6 = 22.5\,\mathrm{km}\,.$$

The segments with highest permitted speeds v_{50} and v_{90} respectively then are

$$s_{50} = s_1 + s_2 + s_3 = v_{50}\left(t_1 + t_3\right) \doteq 16.7\,\mathrm{km}\,,$$
$$s_{90} = s_4 + s_5 + s_6 = v_{50}t_4 + v_{90}\left(t_4 + t_6\right) \doteq 31.8\,\mathrm{km}\,.$$

Now we can start solving the situation with double acceleration. The time of accelerating to v_{50} decreases to

$$t_1' = \frac{v_{50}}{2a} = \frac{1}{2}t_1 = t_2' = 4\,\mathrm{min}\,,$$

which is half of the original time. Similarly, in the case of accelerating from v_{50} to v_{90}

$$t_4' = \frac{v_{90} - v_{50}}{2a} = \frac{1}{2}t_4 = t_5' = 2\,\mathrm{min}\,.$$

We still need to calculate the time of moving with speed v_{50} and the time of moving with speed v_{90} – for this, in both cases we first need to find the distance traversed while accelerating

$$s_1' = s_2' = \frac{1}{2}\frac{v_{50}}{t_1'}{t_1'}^2 = \frac{1}{2}v_{50}t_1' = \frac{1}{4}v_{50}t_1 \doteq 1.7\,\mathrm{km}\,,$$
$$s_4' = s_5' = v_{50}t_4' + \frac{1}{2}\frac{v_{90} - v_{50}}{t_4'}{t_4'}^2 = \frac{1}{2}\left(v_{90} + v_{50}\right)t_4' = \frac{1}{4}\left(v_{90} + v_{50}\right)t_4 \doteq 2.3\,\mathrm{km}\,,$$

and then we calculate the distances and times of moving with uniform speed

$$s_3' = s_1 + s_2 + s_3 - s_1' - s_2' = v_{50}\left(\frac{1}{2}t_1 + t_3\right)\,,$$
$$s_6' = s_4 + s_5 + s_6 - s_4' - s_5' = \frac{1}{2}v_{50}t_4 + v_{90}\left(\frac{1}{2}t_4 + t_6\right)\,,$$
$$t_3' = \frac{s_3'}{v_{50}} = \frac{1}{2}t_1 + t_3 = 16\,\mathrm{min}\,,$$
$$t_6' = \frac{s_6'}{v_{90}} = \frac{1}{2}\frac{v_{50}}{v_{90}}t_4 + \frac{1}{2}t_4 + t_6 \doteq 18.1\,\mathrm{min}\,,$$

The total time is

$$T' = t_0' + t_1' + t_2' + t_3' + t_4' + t_5' + t_6' \doteq 55.1\,\mathrm{min}\,,$$

so the difference which the problem statement is asking for can be calculated as

$$\Delta T = T - T' = \frac{1}{2}\left(t_1 + t_4\left(1 - \frac{v_{50}}{v_{90}}\right)\right) \doteq 4.9\,\mathrm{min}\,.$$

Driving along the same path with double acceleration would save approx. 4.9 min of time, which is approx. 8 %. This only holds under the assumption that Karel is otherwise waiting for equally long times and can freely speed up and slow down.

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Problem BA ... lonely little prince

As the little prince was getting bored on his spherical asteroid B 612, he thought throwing a ball with himself could be fun. He throws the ball from a height of $1.5\,\mathrm{m}$ *above the ground and wants to catch it at the same height after the ball goes around the entire asteroid. How long does one such throw take? Assume that the asteroid B 612 has a radius of* $10\,\mathrm{m}$ *and its density is the same as that of the Earth.* *Terka missed people.*

Since the little prince wants to catch the ball at the same height as the one from which he threw it, we use the formula for circular velocity

$$v_\mathrm{k} = \sqrt{\frac{GM}{r}}\,,$$

where G is the gravitational constant, M is the mass of the asteroid around which the ball orbits at circular velocity, and r is the distance from its center. The mass of the asteroid B 612 can be calculated from the fact that it has the same density as the Earth

$$\begin{aligned}
\rho_\mathrm{p} &= \rho_\mathrm{Z}\,,\\
\frac{m_\mathrm{p}}{V_\mathrm{p}} &= \frac{m_\mathrm{Z}}{V_\mathrm{Z}}\,,\\
m_\mathrm{p} &= m_\mathrm{Z}\frac{\frac{4}{3}\pi r_\mathrm{p}^3}{\frac{4}{3}\pi r_\mathrm{Z}^3}\,,\\
m_\mathrm{p} &= m_\mathrm{Z}\left(\frac{r_\mathrm{p}}{r_\mathrm{Z}}\right)^3\,.
\end{aligned}$$

The last formula that will be useful is the one for the time of one orbit at velocity v along a circular trajectory of radius r

$$t = \frac{2\pi r}{v}\,.$$

After substituting all the known relations into the last formula, we obtain the equation

$$t = \frac{2\pi\left(r_\mathrm{p} + r_0\right)}{\sqrt{\frac{Gm_\mathrm{Z}\left(r_\mathrm{p}/r_\mathrm{Z}\right)^3}{r_\mathrm{p}+r_0}}}\,,$$

where r_0 is the height above the surface from which the little prince throws the ball. This formula can also be simplified to

$$t = \frac{2\pi}{\sqrt{Gm_\mathrm{Z}}}\left(\frac{r_\mathrm{Z}\left(r_\mathrm{p}+r_0\right)}{r_\mathrm{p}}\right)^{\frac{3}{2}}\,.$$

Here we may note that if we were throwing from the surface of the asteroid, the orbital time would be identical for all bodies with the same density, regardless of the radius of the asteroid.

Finally, we just have to plug in the specific values. The time of one orbit of the ball around the little prince's asteroid is

$$t = 6\,251\,\mathrm{s} \doteq 1.7\,\mathrm{h}\,.$$

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Problem BB ... repulsive David

David repels girls. Thus, he decided to charge himself up with a charge of 0.001 C. *What charge would a girl need to have for the electric force to be greater than David's repulsive force, which is inversely proportional to the square of the distance with the proportionality constant* $160\,\mathrm{N{\cdot}m^2}$*?*

David can't find a girlfriend.

Since David's repulsive force is inversely proportional to the distance, we can write it in the form $F_{\text{David}} = \varphi/r^2$, where φ is a constant from the problem statement. We are interested in when the equality of forces $F_{\text{David}} = F_C$ occurs. Let's expand the formula as follows

$$\frac{\varphi}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 \cdot q_2|}{r^2}\,.$$

We need to calculate q_2, and since equal charges repel each other, q_2 will be negative. This allows us to drop the absolute value and adjust the formula to

$$q_2 = -\frac{4\pi\varepsilon_0\varphi}{q_1} = -1.78 \cdot 10^{-5}\,\mathrm{C}\,.$$

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Problem BC ... student's

What is the minimum necessary length of copper wire, in order to be able to use it for heating? The socket has an AC voltage of 230 V, *and the cross-section of the wire is* $5\,\mathrm{mm^2}$. *Our circuit breakers carry a maximum current of* 6 A. *The specific electrical resistance of copper is* $0.0178\,\Omega{\cdot}\mathrm{mm^2{\cdot}m^{-1}}$.

Lukáš complained about his room being cold.

First, let's calculate the resistance at which the current 6 A will flow. Using Ohm's law, we get

$$R = \frac{U}{I}\,.$$

Next, let's determine how long the wire must be to have the required resistance

$$R = \rho\frac{l}{S}\,,$$

where ρ is the specific electrical resistance of copper, S is the cross-section of the wire and l is its length. By comparing the two equations, we obtain

$$\rho\frac{l}{S} = \frac{U}{I}\,.$$

From the relation above, let's express l and plug in the given values

$$l = \frac{S}{\rho}\frac{U}{I} = 10.8\,\mathrm{km}\,.$$

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Problem BD ... spring warm-up

Danka and Dano utilize a straightforward exercise machine designed to strengthen their arms – consisting of two handles (of negligible length) connected by parallel springs. The number of springs is easily adjustable. All the springs employed possess a stiffness of k and a rest length of l_0. When Danka applies a force F and uses two springs in the machine, she elongates the machine to a total length of l_1. To what overall length does the stronger Dano stretch the machine if he incorporates three springs into the apparatus and applies a force of $2F$? Express the result solely in terms of l_0 and l_1. *Karel created a problem to exercise.*

The general relationship between a force F acting on a spring (with stiffness k) elongated by Δl is

$$\Delta l = \frac{F}{k}\,.$$

When connecting the springs in parallel, the force gets distributed between them. Consequently, when we extend all springs of the same stiffness by the same length, the force is evenly distributed, and their stiffness effectively combines. In other words, for the elongation in Danka's case, we can write

$$\Delta l_1 = \frac{F}{2k}\,,$$

and in Dano's case

$$\Delta l_2 = \frac{2F}{3k}\,.$$

Nevertheless, the problem statement specifies the exclusive utilization of the rest length l_0 and the elongation of the spring by Danka denoted as l_1. For the latter

$$l_1 = l_0 + \Delta l_1 = l_0 + \frac{1}{2}\frac{F}{k}\,.$$

Subsequently, we want to know the total length of Dan's spring, i.e., the value of

$$l_2 = l_0 + \Delta l_2 = l_0 + \frac{2}{3}\frac{F}{k}\,.$$

To get the answer in the required lengths, we express F/k from the relation for l_1 and add it to the relation for l_2

$$l_2 = l_0 + \frac{2}{3}\cdot 2\left(l_1 - l_0\right)\,.$$

After a slight simplification, we get the answer that the total length of the springs throughout Dano's exercise is

$$l_2 = \frac{4}{3}l_1 - \frac{1}{3}l_0\,.$$

Finally, let us note that, just as the problem required, for expressing the result we do not need to be acquainted with the absolute stiffness of either spring – it was sufficient to know that the stiffness were the same.

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Problem BE ... aircraft tow tractors

The airport has tow tractors that can move large airplanes weighing up to M. The mass of a tow tractor is m, and the friction between its wheels and the runway is f. What is the maximum acceleration a tow tractor can have without slipping if it is pulling an airplane behind it?

Reportedly, at an airport in Brno, tractors are used to speed up planes.

Consider that the tow tractor can exert a total force F on itself and the aircraft. Then, according to Newton's second law, they will both move with an acceleration of

$$a = \frac{F}{M+m}\,.$$

Now let's see what this maximum horizontal force is. From the third Newton's law, the force F must arise in response to some other horizontal force acting between the tractor and the runway. The only such force is the friction between the wheels of the tow tractor and the runway surface. The maximum magnitude of the frictional force is given as $F_t = fN$, where N is the normal force acting on the tractor perpendicular to the ground and f is the coefficient of friction. In this case, the normal force corresponds to the gravitational force acting on the tractor, i.e. $N = mg$. Overall, we obtain

$$F_\mathrm{t} = fN = fmg = F \quad \Rightarrow \quad a = \frac{F_\mathrm{t}}{M+m} = \frac{mgf}{M+m}\,.$$

If the tractor could spin its wheels with greater angular acceleration than what we found, it would start to slide down the runway, and the frictional force would not increase, so it would not change our result.

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Problem BF ... boredom on the subway

On the way home from school, Pepa got bored on the subway, so he pulled out a mathematical pendulum from his pocket. It had a 1.002 m long rod with a 1.103 kg weight on its end, and he hung it from the ceiling of the subway car. Now, considering the subway's acceleration of $2.350\,\mathrm{m\cdot s^{-2}}$, how much larger is the period of the small oscillations of the mathematical pendulum when the subway is in motion compared to its period when stationary?

Pepa was really bored on the subway.

The pendulum is subjected to gravitational acceleration g in the vertical direction and an inertial acceleration a in the horizontal direction. The total acceleration will be the sum of their vectors a', which, due to their perpendicularity, can be calculated simply using the Pythagorean theorem

$$a' = \sqrt{g^2 + a^2}\,.$$

The period of the small oscillations is determined from the well-known formula for the period of a mathematical pendulum

$$T = 2\pi\sqrt{\frac{l}{g}}\,.$$

However, instead of g, we have to substitute the acceleration a', and obtain the result

$$T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a^2}}}\,.$$

From there

$$T_2 - T_1 \doteq -0.028\,\mathrm{s}\,.$$

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Problem BG ... Titanic

A metal boat with two passengers has a total weight of $m = 210\,\mathrm{kg}$. It is floating on the pond surface and has the shape of a rectangular cuboid of dimensions $2.0\,\mathrm{m} \times 0.8\,\mathrm{m} \times 0.5\,\mathrm{m}$ (the shortest dimension is vertical). Suddenly, a crocodile attacks the boat, and a crack of cross-section $A = 10\,\mathrm{cm}^2$ appears at the middle of the bottom of the vessel. Therefore, water starts flowing inside. How much time remains for the passengers until every part of the boat disappears below the surface? *Jarda has seen a crocodile in Brno.*

Solution through Bernoulli and Archimedes

The water is streaming inward at a velocity of $v = \sqrt{2g\Delta h}$, where $\Delta h = h_2 - h_1$ represents the difference in height between the pond surface and the water level inside the boat. The height of the water in the boat is h_1, and the bottom of the rowboat is h_2 below the pond surface.

Using Archimedes' principle, we determine the dependence of h_2 on the water mass in the boat. Gravity exerts a downward force, expressed mathematically as $F_\mathrm{g} = (m + V\rho)g = (m + Sh_1\rho)g$, where V represents the volume of water in the boat, ρ is the density of the water, and $S = 1.6\,\mathrm{m}^2$ is the area of the boat base. It is compensated by buoyancy of magnitude $F_\mathrm{v} = Sh_2\rho g$.

Thus,

$$(m + Sh_1\rho)\,g = Sh_2\rho g \quad\Rightarrow\quad \delta h = h_2 - h_1 = \frac{m}{S\rho}\,.$$

The speed of water flowing into the boat is constant. The boat disappears exactly when $h_2 = c = 0.5\,\mathrm{m}$, i.e. when the water flows inside from the top. That happens when the water surface in the boat is at a height

$$h_1 = c - \delta h = c - \frac{m}{S\rho}\,.$$

The volume of the water in the boat at that moment $Sh_1 = Avt$ is equal to the product of constant volumetric flow rate and time. From that, we can express the time as

$$t = \frac{Sh_1}{Av} = \frac{abc - \frac{m}{\rho}}{A\sqrt{2g\frac{m}{ab\rho}}} = 367\,\mathrm{s}\,.$$

Solution purely through the law of conservation of energy

How does the energy throughout the sinking process change? The potential energy of the boat decreases and transforms mainly to the kinetic energy of the water flowing inside. As the water enters the vessel, the potential energy will convert into heat; however, that shall not concern us. Hence, we will rely on the equivalence between the change in the boat's potential energy and the water's kinetic energy.

Let us denote the rate of the boat's sinking v_L. Then, in time $\mathrm{d}t$ the potential energy of the vessel decreases by $\mathrm{d}E_\mathrm{p} = -mgv_\mathrm{L}\,\mathrm{d}t$. At the same time, due to this displacement, the boat reaches a location previously occupied by water, corresponding to a volume of $\mathrm{d}V = Sv_\mathrm{L}\,\mathrm{d}t$. This water enters the boat through a hole A in time $\mathrm{d}t$. Thus, it must have the velocity

$$v = \frac{\mathrm{d}V}{A\,\mathrm{d}t} = \frac{S}{A}v_\mathrm{L}\,,$$

which makes sense.

Therefore, the kinetic energy of the water increases in time $\mathrm{d}t$ by

$$\mathrm{d}E_\mathrm{k} = \frac{1}{2}\,\mathrm{d}V\rho v^2 = \frac{1}{2}A\rho v^3\,\mathrm{d}t\,.$$

We lay the differences of energies equal and express the rate of the water flowing inside the boat v

$$\begin{aligned}\mathrm{d}E_\mathrm{k} &= -\,\mathrm{d}E_\mathrm{p}\\ \frac{1}{2}A\rho v^3\,\mathrm{d}t &= mgv\frac{A}{S}\,\mathrm{d}t\\ v &= \sqrt{\frac{2mg}{\rho S}}\,,\end{aligned}$$

which is the same as the solution through Bernoulli and Archimedes. We can also determine the rate of the boat's sinking $v_\mathrm{L} = vA/S$ and subsequently calculate the time of the sinking of the rowboat, just as in the previous solution, and we again get $t = 367\,\mathrm{s}$.

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Problem BH ... pulleys with mass rope

Consider a pulley with a negligible radius, across which we suspend a rope of length L. A rectangular cuboid with a weight m_1 hangs on one of its ends, while a rectangular cuboid with a weight m_2 on the other. What will be the acceleration of cuboid m_1 when the system is released? In the moment of release, the rope lays symmetrical to the pulley. The rope has a uniform linear density and an overall mass of m. Neglect both friction and the weight of the pulley.

Lego eventually decided to submit something like this.

The intuitive solution

As the rope is positioned symmetrically, its gravity will not contribute to any acceleration (the components from both parts of the rope will cancel each other out). Therefore, our focus will be solely on the inertial effect of its mass: when the entire system is set into motion, it will experience an additional weight of m.

If the rope were massless, the acceleration would be

$$a = g\frac{m_1 - m_2}{m_1 + m_2}\,,$$

but since we need to accelerate a mass greater by m, then the acceleration must be

$$a = g\frac{m_1 - m_2}{m_1 + m_2 + m}\,.$$

Another intuitive perspective on this outcome is that we effectively added the mass $m/2$ to both sides, given the symmetrical distribution of the rope. This addition cancels out in the numerator and aggregates to m in the denominator. Thus, the result aligns logically with our understanding. Task accomplished.

Proper solution

Each rope element is subject to three forces: a gravitational force of magnitude $\mathrm{d}mg$, a tensile force from the rest of the rope in one direction, and a tensile force from the rest of the rope in the other direction (or a tensile force from two adjacent elements). Derived from Newton's third law, we deduce that elements within the rope experience forces of equal magnitude but opposite directions from their neighboring elements. Consequently, introducing a variable "tension" $T(x)$, dependent on the position in the rope, becomes meaningful. This variable informs us about the extent to which rope elements are being pulled at a specific point. Furthermore, if a rope element has a length $\mathrm{d}x$, employing a first-order approximation allows us to compute the resultant force exerted by the remaining portion of the rope on that particular element:

$$\mathrm{d}F_T(x) = T(x + \mathrm{d}x) - T(x) = T(x) + \mathrm{d}x\frac{\mathrm{d}T(x)}{\mathrm{d}x} - T(x) = \mathrm{d}x\frac{\mathrm{d}T(x)}{\mathrm{d}x}\,.$$

The rope has a homogeneous length density, i.e., there is a linear relation $\mathrm{d}m = \mathrm{d}xm/L$ between the length of the element $\mathrm{d}x$ and the mass of the element $\mathrm{d}m$.

For every element, the equation of motion must be satisfied, meaning that the total force acting on it should be equal to the product of its mass and acceleration.

$$\begin{aligned}
\mathrm{d}ma &= \pm\,\mathrm{d}mg + \mathrm{d}x\frac{\mathrm{d}T(x)}{\mathrm{d}x}\\
\mathrm{d}x\frac{m}{L}a \pm \mathrm{d}x\frac{m}{L}g &= \mathrm{d}x\frac{\mathrm{d}T(x)}{\mathrm{d}x}\\
\frac{m}{L}(a \pm g) &= \frac{\mathrm{d}T(x)}{\mathrm{d}x}\,,
\end{aligned}$$

We have derived a differential equation for the tension $T(x)$ as a function of the position in the rope x, where $x = 0$ is the point of contact with weight m_2, and $x = L$ corresponds to the weight m_1. The entire rope experiences the same acceleration, whose direction gets reversed at the midpoint. However, from the rope's perspective, it continues in the same direction (either towards increasing or decreasing x), making a effectively a constant. Specifically, a positive a implies acceleration in the direction of increasing x, corresponding to the scenario where m_1 accelerates downward.

Regarding gravity, before it is $\pm$ in the equation because in the half where $x < L/2$, gravity pulls in the direction of decreasing x, and in the other half ($x > L/2$), it pulls in the direction of increasing x. To express this more precisely, we can replace $\pm$ in the last equation with $\operatorname{sgn}(L/2 - x)$ (and in the first equation, the opposite holds).

Let's designate the force with which the rope pulls the weight m_2 as T_0, while $T(0) = T_0$. In the first half of the rope

$$\frac{m}{L}(a+g) = \frac{\mathrm{d}T(x)}{\mathrm{d}x}\,,$$

so the tension in it will change as a linear function

$$T(x) = T_0 + \frac{m}{L}(a+g)x\,,$$

so at the point where the rope rotates, $T(L/2) = T_0 + (m/2)(a+g)$ holds.

At the other half of the rope

$$\frac{m}{L}(a-g) = \frac{\mathrm{d}T(x)}{\mathrm{d}x}\,,$$

so we get the following for the tension

$$T(x) = T\left(\frac{L}{2}\right) + \frac{m}{L}(a-g)\left(x - \frac{L}{2}\right) = T_0 + \frac{m}{2}(a+g) + \frac{m}{L}(a-g)\left(x - \frac{L}{2}\right)\,.$$

Thus, at the point of contact with the block of the mass m_1

$$T(L) = T_0 + \frac{m}{2}(a+g) + \frac{m}{2}(a-g) = T_0 + ma\,,$$

where $T(L)$ is the force with which the rope pulls the cuboid m_1 upwards.

We can write equations of motion for the blocks. We will continue to denote a as the acceleration by which m_1 accelerates downward and hence m_2 accelerates upward. Thus, the equation of motion for m_2 will be

$$m_2 a = T_0 - m_2 g\,.$$

We can express the unknown T_0 from this equation as $T_0 = m_2(a+g)$. We write the equation of motion for m_1 and substitute

$$\begin{aligned}
m_1 a &= m_1 g - (T_0 + ma)\,,\\
m_1 a &= m_1 g - m_2(a+g) - ma\,,\\
(m_1 + m_2 + m)a &= (m_1 - m_2)g\,.\\
a &= g\frac{m_1 - m_2}{m_1 + m_2 + m}\,.
\end{aligned}$$

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Problem CA ... turbomolecular pump

Turbomolecular pumps, designed to achieve low pressures, operate by altering the momentum of gas particles in the direction of the pumped volume. For this process to be effective, the blades of the pump's rotor must rotate at speeds comparable to the thermal speed of the gas molecules. Let's consider a rotor with a diameter of $d = 15\,\mathrm{cm}$ *and nitrogen as the pumped gas with a temperature of* $25\,\mathrm{^\circ C}$. *At what frequency (not angular frequency) must the rotor in the pump rotate to make the ends of the blades move at the root-mean-square speed of nitrogen particles at the given temperature? The molar mass of nitrogen is* $M_{\mathrm{N}_2} = 28\,\mathrm{g{\cdot}mol^{-1}}$.

Jarda was studying for an exam in vacuum physics.

The root-mean-square speed of nitrogen gas molecules is given by

$$v = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M_{\mathrm{N}_2}}} = 515\,\mathrm{m{\cdot}s^{-1}}$$

where m is the mass of one molecule of nitrogen, k the Boltzmann constant, R the molar gas constant, T is the thermodynamic temperature and M_{N_2} the molar mass of nitrogen given in the problem statement.

We express the relationship between the velocity of the blades at the circumference and the frequency of rotation

$$v = \omega r = 2\pi f \frac{d}{2} = \pi d f\,,$$

where $\omega = 2\pi f$ is the angular frequency of rotation and $r = d/2$ is the radius of the rotor. Expressing the desired frequency,

$$f = \sqrt{\frac{3RT}{M_{\mathrm{N}_2}}}\frac{1}{\pi d} \doteq 1\,090\,\mathrm{Hz}\,.$$

This result aligns with frequencies commonly used in laboratories. Thanks to turbomolecular pumps, it is possible to reach pressures as low as $1 \cdot 10^{-9}\,\mathrm{Pa}$. However, these pumps require pre-pumping with another type of pump. Due to their high rotational speeds, precise balancing is crucial. On the positive side, no oil lubricants are needed, which eliminates the risk of contamination inside the vacuum apparatus.

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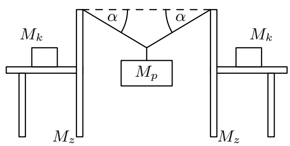
Problem CB ... sliding dryer



Terka made a DIY clothes dryer in her dorm room by positioning two chairs with their backs turned toward each other and extending a piece of wool twine between them. The laundry hangs at an angle of $\alpha = 13^\circ$. *However, the chairs began to shuffle towards each other, so Terka loaded them with canned beans in chili sauce. What is the minimum amount of canned food she would need to keep the chairs in place? The weight of the chair is* $M_{\mathrm{z}} = 4.8\,\mathrm{kg}$, *the weight of the canned food* $M_{\mathrm{k}} = 480\,\mathrm{g}$ *and the coefficient of static friction between the chair and the floor is* $f = 0.65$. *We can approximate laundry by a mass point of weight* $M_{\mathrm{p}} = 2\,\mathrm{kg}$ *suspended in the middle of the clothesline.*

Terka observed Terka hanging the laundry.

We will address the problem by analyzing the distribution of the gravitational force acting on the laundry, denoted as F_G. As the laundry is at rest, a tensile force must act from the direction above the hinge and be in equilibrium with the gravitational force. From the distribution of forces and knowing the angle of deflection, we can express this tensile force F_t as

$$\sin\alpha = \frac{\frac{F_\mathrm{G}}{2}}{F_\mathrm{t}}$$
$$F_\mathrm{t} = \frac{F_\mathrm{G}}{2\sin\alpha}\,.$$

We must be careful that the tensile force is applied to the laundry twice, each time from one of the chairs, so we have to divide the pulling force in the calculation by two.

The same tensile force, just in the opposite direction, will act on the chair. We will be interested in its horizontal (F_v) and vertical (F_s) components. These can be expressed again from the distribution of the forces

$$F_\mathrm{v} = F_\mathrm{t}\cos\alpha = \frac{F_\mathrm{G}}{2}\frac{\cos\alpha}{\sin\alpha} = \frac{F_\mathrm{G}}{2}\cot\alpha\,,$$
$$F_\mathrm{s} = F_\mathrm{t}\sin\alpha = \frac{F_\mathrm{G}}{2}\,.$$

At the same time, the two chairs with the cans will exert a total gravitational force on the ground

$$F_\mathrm{c} = g\left(2M_\mathrm{z} + xM_\mathrm{k}\right)\,,$$

where x is the number of cans. In the following calculations, we will work with just half the value as we are solving the whole situation for one-half of the system, i.e., one chair and half of the cans.

The frictional force must be equivalent to the horizontal force acting on the chair to prevent it from sliding. Therefore, we get the condition

$$f\left(\frac{F_\mathrm{c}}{2} + F_\mathrm{s}\right) = F_\mathrm{v}\,,$$
$$\frac{fg\left(2M_\mathrm{z} + xM_\mathrm{k}\right)}{2} + \frac{fgM_\mathrm{p}}{2} = \frac{gM_\mathrm{p}}{2}\cot\alpha\,,$$

taking advantage of the fact that $F_\mathrm{G} = gM_\mathrm{p}$. We can further multiply the equation by two, reduce g, and use it to express the number of cans we are looking for

$$f\left(2M_\mathrm{z} + xM_\mathrm{k}\right) + fM_\mathrm{p} = M_\mathrm{p}\cot\alpha\,,$$
$$xM_\mathrm{k} = \frac{M_\mathrm{p}\cot\alpha}{f} - M_\mathrm{p} - 2M_\mathrm{z}\,,$$
$$x = \frac{M_\mathrm{p}\cot\alpha - M_\mathrm{p}f - 2M_\mathrm{z}f}{fM_\mathrm{k}}\,.$$

So after the substitution, we get the result $x = 3.6$, and the correct answer is that Terka needs at least four cans.

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Problem CC ... a magical bath

Jindra has a bath in which the water level decreases at a constant rate of v_0 independently from its height h over the outlet hole. The outlet hole has a cross-section S_0. You can assume that the approximation $gh \gg v_0^2$ applies. Determine the cross-section of the bath as a function of height $S = S(h)$. *Jindra played with boats.*

Mathematically, Bernoulli's principle determines the motion of fluids as

$$\frac{1}{2}\rho v^2 + p + \rho g h = \mathrm{const},$$

where ρ is the density of the fluid, v is its velocity at a given point, p is the pressure at that point due to the external environment, $g = 9.81\,\mathrm{m\cdot s^{-2}}$ is the gravitational acceleration and h is the height of the point relative to the reference frame.

The water in Jindra's bath is a fluid satisfying Bernoulli's equation. All heights, denoted as h, will be referenced to the outlet. At the water surface in the bath, where the water's velocity v_0 remains constant (as per the information provided in the problem statement), the height of the water above the outlet is h, and the pressure equals the atmospheric pressure $p = p_\mathrm{a}$. At the outlet hole, water exits with an unknown velocity v, the height above the outlet hole is zero, and the pressure equals the atmospheric pressure $p = p_\mathrm{a}$. Thus, we get the equation

$$\frac{1}{2}\rho v_0^2 + p_\mathrm{a} + \rho g h = \frac{1}{2}\rho v^2 + p_\mathrm{a}\,,$$

$$\frac{1}{2}v_0^2 + gh = \frac{1}{2}v^2\,.$$

Now, we introduce another equation into the scenario – the continuity equation. If at one point water flows through a hole of cross-section S_1 at a velocity v_1 and at another point, it flows through a hole of cross-section S_2 at a velocity v_2, then the volume flow rate q satisfies

$$q = S_1 v_1 = S_2 v_2 = \mathrm{const}\,.$$

We get a set of two equations with two unknowns S and v from Bernoulli's principle and the continuity equation

$$S v_0 = S_0 v\,,$$

$$\frac{1}{2}v_0^2 + gh = \frac{1}{2}v^2\,.$$

We want to express the dependence of the cross-section of the bath S on the height above the outlet h. We express the velocity v from the second equation

$$v = \sqrt{v_0^2 + 2gh}$$

and substitute it into the first equation

$$S v_0 = S_0 \sqrt{v_0^2 + 2gh}\,.$$

The dependence of the cross section $S(h)$ of the bath on the height h above the outlet opening is

$$S(h) = \frac{S_0}{v_0}\sqrt{v_0^2 + 2gh} = S_0\sqrt{1 + \frac{2gh}{v_0^2}} \approx S_0\sqrt{\frac{2gh}{v_0^2}}\,.$$

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Problem CD ... clash of the titans

Traffic is so heavy on the highway that the vehicles maintain a time separation of $\tau = 3\,\mathrm{s}$. Nonetheless, one truck decides to overtake another truck that has a large gap in front of it. It moves into the left lane and accelerates to $v_\mathrm{L} = 95\,\mathrm{km{\cdot}h^{-1}}$, while the overtaken truck is still going $v_\mathrm{P} = 90\,\mathrm{km{\cdot}h^{-1}}$. The cars in the left lane are going $v_\mathrm{a} = 125\,\mathrm{km{\cdot}h^{-1}}$, but when they approach the vehicle in front of them at a distance of τv_L, they immediately slow down to a speed of v_L. How many cars will have to slow down like this? The length of both trucks is $L = 15\,\mathrm{m}$ and the overtaking truck will merge into the original lane while maintaining the same distance as before overtaking. *Jarda heard about a long traffic jam on the highway.*

When the truck moves into the left lane, its spacing behind the other truck is τv_P. To overtake the other truck, it needs to get the same distance ahead of him while traveling the length of the overtaken truck and its own. Thus, in the left lane, he spends time

$$T = \frac{2\tau v_\mathrm{p} + 2L}{v_\mathrm{L} - v_\mathrm{P}} \doteq 130\,\mathrm{s}\,.$$

The gap between cars in the left lane is τv_a, which can now be shortened to τv_L. Each car can therefore continue at speed v_a until it gets within the distance v_L of the truck in the left lane and slows down immediately to v_L. This maneuver takes each car the time $t = \tau\left(v_\mathrm{a} - v_\mathrm{L}\right) / \left(v_\mathrm{a} - v_\mathrm{L}\right) = \tau$. The number of cars that have to slow down is then

$$N = \frac{T}{\tau} = \frac{2\tau v_\mathrm{P} + 2L}{\left(v_\mathrm{L} - v_\mathrm{P}\right)\tau} \doteq 43\,.$$

In such a situation, a convoy of about 43 vehicles, which had to slow down to the speed v_L, would form behind the overtaking truck.

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Problem CE ... charged square

Marbles with charge q are placed in the vertices of a square, in the center of which is a marble with charge $Q = -kq$. What must be the value of the constant k for the whole system to be in equilibrium? *Danka played with marbles.*

From the symmetry of the problem, it is only necessary for the forces acting on the marbles to be in equilibrium in the vertices of the square – the forces acting on the marble in the center will always cancel out. Moreover, it is only necessary to consider the marble at one vertex – the situation will be the same for all the others. The repulsive forces from the marbles in the vertices will be compensated by the attractive force from the one in the center of the square. So, the following holds

$$\sqrt{\left(\frac{1}{4\pi\varepsilon}\frac{q^2}{a^2}\right)^2 + \left(\frac{1}{4\pi\varepsilon}\frac{q^2}{a^2}\right)^2} + \frac{1}{4\pi\varepsilon}\frac{q^2}{2a^2} = \frac{1}{4\pi\varepsilon}\frac{kq^2}{a^2/2}\,,$$

from where we can easily derive that

$$k = \left(\frac{1 + 2\sqrt{2}}{4}\right) \doteq 0.957\,.$$

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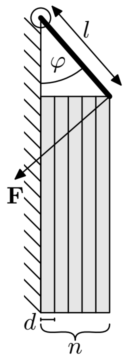
Problem CF ... a folder with a clip



Let us consider a paper folder equipped with a clip on its upper side to secure papers. The clip, with a length of 2 cm*, operates on a spring with a radial stiffness of* 1.0 Nm·rad^{-1}*. As papers are added between the folder and the clip, forming a rectangular cuboid pinned down at the edge, coefficient of friction f exists between the papers and the clip, as well as between the papers and the folder. Infinite friction is assumed between the papers themselves. We found out that all the papers fall out once we add* 130 *under the clip and turn the folder vertically. What is the value of the coefficient f? Consider the width of a single sheet of paper* 0.1 mm *and its weight* 5.5 g.

Jarda carries a handy folder such as this one to school.

Let us designate the length of the clip as l, the radial stiffness of the spring as c, the width of a single sheet of paper as d, its weight as m, and the number of papers as n. The torsion applied to the clip by the spring can be determined as

$$M = c\varphi \,,$$

where φ is the angle of rotation of the clip compared to the folder. We can calculate the size of the angle from the number of papers as

$$\sin\varphi = \frac{nd}{l} \,.$$

Therefore, the force applied to the paper is equal to $F = M/l$ and points perpendicularly to the plane of the clip. We will decompose this force into the component perpendicular to the plane of the folder and the one parallel to it. Once we turn the folder perpendicularly to the ground, the parallel component points downwards, while the perpendicular component points into the paper. The gravitational force of the paper bundle $F_{\mathrm{g}} = nmg$, where g is the gravitational acceleration, points downwards as well.

Against the forces pointed downwards, friction forces are applied between the clip and the papers, and the papers and the folder. Because the papers are moving neither toward the folder nor into the clip, the normal forces applying to them are in balance. The normal force from the clip

$$N = F\cos\varphi$$

is compensated by the reaction force of the folder of the same size. We will calculate the friction force keeping the paper stationary as

$$F_{\mathrm{t}} = 2fN = 2fF\cos\varphi \,,$$

where f is the coefficient of the friction from the problem statement and the numerical coefficient 2 comes from both of the normal forces being applied at the same time. This force must be in balance with the forces

$$F_{\mathrm{g}} + F\sin\varphi = nmg + F\sin\varphi = 2fF\cos\varphi \,.$$

Substituting for F we get an equation

$$nmgl + c\varphi\sin\varphi = 2fc\varphi\cos\varphi \,,$$

from which, after substituting for the angle, we can express the friction coefficient as

$$f = \frac{nmgl + c\varphi\sin\varphi}{2c\varphi\cos\varphi} = \frac{nmgl + c\arcsin\left(\frac{nd}{l}\right)\frac{nd}{l}}{2c\arcsin\left(\frac{nd}{l}\right)\sqrt{1-\left(\frac{nd}{l}\right)^2}} = 0.56\,.$$

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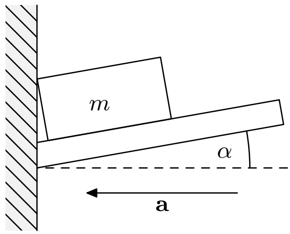
Problem CG ... a falling suitcase



Above the seats in the train on a shelf tilted by an angle $\alpha = 10^\circ$ is placed a wooden suitcase with the weight $m = 10\,\mathrm{kg}$. With what minimal acceleration must the train move in order for the luggage to slide from the shelf? The static friction coefficient between the suitcase and the shelf is $f = 0.4$.

Train rides are unpredictable.

The key to solving this problem is to devise an accurate representation of the forces acting upon the suitcase and distribute them as forces perpendicular to the shelf ($F_\perp$) and parallel to the shelf $\left(F_\parallel\right)$. There will be three forces acting upon the luggage: gravitational ($F_\mathrm{G} = mg$), friction force ($F_\mathrm{T} = ma$), and inertia (F_S) caused by the train accelerating. We can express the gravity and inertia components as

$$\begin{aligned} F_{\mathrm{G}\perp} &= mg\cos\alpha\,,\\ F_{\mathrm{G}\parallel} &= mg\sin\alpha\,,\\ F_{\mathrm{S}\perp} &= ma\sin\alpha\,,\\ F_{\mathrm{S}\parallel} &= ma\cos\alpha\,. \end{aligned}$$

We can determine the friction force as a sum of forces perpendicular to the shelf multiplied by the static friction coefficient, thus

$$F_\mathrm{T} = f\left(F_{\mathrm{G}\perp} + F_{\mathrm{S}\perp}\right) = fmg\cos\alpha + fma\sin\alpha\,.$$

For the suitcase to move forward, the result of forces must be equal to zero (more precisely, a bit greater and oriented in the same direction as $F_{\mathrm{S}\parallel}$). As forces are vector quantities, we must also take into account their directions

$$\begin{aligned} F_{\mathrm{S}\parallel} &= F_\mathrm{T} + F_{\mathrm{G}\parallel}\\ ma\cos\alpha &= fmg\cos\alpha + fma\sin\alpha + mg\sin\alpha\\ a &= \frac{fmg\cos\alpha + mg\sin\alpha}{m\cos\alpha - fm\sin\alpha} \end{aligned}$$

and after quantification

$$a \doteq 6.1\,\mathrm{m{\cdot}s^{-2}}\,.$$

We observe that it is approximately two-thirds of the gravitational acceleration; hence, it is improbable for the train to move with such an acceleration. An exception could be a traffic accident or some other kind of crisis.

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Problem CH ... dioptric

Hanka wears dioptric glasses with optical power $-0.5\,\mathrm{D}$ *to school. Once, however, she forgot them at home so she needed to sit at a different spot than usual. What is the maximum distance from the blackboard at which she can sit in order to see what is written on the blackboard sharply?* *Hanka was playing quidditch with glasses.*

Since Hanka wears glasses with negative dioptres, which are concave. It means that she cannot see clearly at greater distances – she is nearsighted. Usually, a human without this disability has the so-called *far point* (the furthest point which can be seen sharply) at infinity. For a nearsighted human, the far point is closer and it is impossible to focus at further points. Therefore, glasses with concave lenses are needed in order to create a virtual image somewhat closer.

If the source of the image Hanka sees is at infinity, rays from it are parallel to the optical axis. By definition, a concave lens projects the source into its focus, so it drags the image closer from infinity. The virtual image of any point closer than infinity lies between the concave lens and its image focal point. Therefore, if Hanka wears glasses with which she can see even points at infinity sharply, then she will certainly see all closer points sharply too.

The distance of Hanka's far point should be the same as the focal length of her glasses, so that she could see even points at infinity sharply. If the distance of her far point was different, she would need to wear different glasses. Now, we just need to remember the relation between the optical power and focal length

$$f = \frac{1}{D} = \frac{1}{-0.5\,\mathrm{D}} = -2\,\mathrm{m}\,,$$

where the negative sign means that the far point is on the opposite side of the lens than the eye, as we assumed. The far point at the distance $2\,\mathrm{m}$ is the furthest point which Hanka sees sharply, so she has to sit at most $2\,\mathrm{m}$ from the blackboard.

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Problem DA ... combustion

During the combustion of gasoline, two molecules of $\mathrm{C_8H_{18}}$ *along with* $25\,\mathrm{O_2}$ *enter the reaction to form carbon dioxide and water. Consider the fuel consumption of a car* $6\,\mathrm{l}\cdot(100\,\mathrm{km})^{-1}$ *and the density of gasoline* $755\,\mathrm{kg{\cdot}m^{-3}}$*. Under normal conditions, what volume of* $\mathrm{CO_2}$ *is produced during the car's journey from Brno to Prague along a* $207\,\mathrm{km}$ *long route?*

One day, Jarda will buy a Hydrogen-powered car.

One liter of gasoline contains the following amount of substance of carbon

$$n_{\mathrm{C}} = 8\frac{V\rho}{M_{\mathrm{C_8H_{18}}}} \doteq 53\,\mathrm{mol}\,,$$

where $V = 1\,\mathrm{l}$, ρ is the density of gasoline and $M_{C_8H_{18}} \doteq 114\,\mathrm{g{\cdot}mol^{-1}}$ is its molar mass. Since for every mole of carbon, one mole of carbon dioxide is produced, we then also know the amount of substance of CO_2.

Substituting into the equation of state of the ideal gas gives the volume V as

$$V_{CO_2} = \frac{n_C R T_a}{p_a} = 8\frac{V\rho}{M_{C_8H_{18}}}\frac{RT_a}{p_a} \doteq 1.3\,\mathrm{m^3}\,,$$

where R is the gas constant, T_a is the normal temperature and p_a is the atmospheric pressure. The car has a fuel consumption of $6\,\mathrm{l}\cdot(100\,\mathrm{km})^{-1}$, so in a trip of $207\,\mathrm{km}$ it burns

$$\frac{6\,\mathrm{l}}{100\,\mathrm{km}} \cdot 207\,\mathrm{km} = 12.4\,\mathrm{l}$$

of gasoline. This corresponds to releasing $12.4\,\mathrm{l} \cdot 1.3\,\mathrm{m^3{\cdot}l^{-1}} = 16\,\mathrm{m^3}$ into the air. However, the gasoline we normally use in our vehicles is not the pure substance with the formula above, so various filters and catalysts must be used in the exhaust system to ensure that as few hazardous substances as possible are released into the air.

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Problem DB ... a pond on a mirror

Jindra owns a Newtonian telescope with a hollow spherical primary mirror. It has a radius of curvature $r = 2.40\,\mathrm{m}$. Once Jindra left the telescope uncovered in a vertical position outside, the telescope was rained on, and the primary mirror was filled with water with a refractive index of $n = 1.33$. How many times did the focal length of the primary mirror decrease? Neglect the thickness of the water layer compared to the focal length.

Jindra also found diving beetles in a telescope tube.

The definition of a focal point is a point on the optical axis where reflected or refracted rays arriving parallel to the optical axis intersect. The focal length is then the distance of the focal point from the top of the mirror (the point on the mirror's surface lying on the optical axis). In the paraxial approximation, we assume that the angles of all incoming and reflected light rays with the optical axis are small $\alpha \ll 1$. Rays parallel to the optical axis hitting the mirror surface at perpendicular distance h from the optical axis will be reflected at an angle of $2\alpha \approx 2h/r$. The paraxial approximation for a spherical mirror with a radius of curvature r gives the focal length

$$f_0 = \frac{h}{2\alpha} = \frac{r\alpha}{2\alpha} = \frac{r}{2}\,.$$

This relationship holds for a spherical mirror without water inside.

Rays parallel to the optical axis will pass unchanged through the water layer on the mirror. They are then reflected from the surface of the mirror according to the law of reflection. Let us denote the angle *point of reflection – the center of curvature – the peak of the mirror* as α. The ray is then reflected at an angle of 2α with respect to the optical axis. However, it still has to pass through the water-air interface. According to Snell's law of refraction, it breaks at an angle of β towards the optical axis

$$\sin(\beta) = n\sin(2\alpha)\,,$$
$$\beta \approx 2n\alpha\,.$$

In the derivation, we used the approximation for small angles $\sin x \approx x$ for $|x| \ll 1$. The new focal length is

$$f = \frac{h}{\beta} = \frac{r\alpha}{\beta} = \frac{r}{2n} = \frac{f_0}{n} = \frac{f_0}{1.33}\,.$$

So, the focal length has decreased 1.33 times.

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Problem DC ... wombat's

The tiny wombat Cooper cannot produce enough heat to cover the heat loss to the burrow's surroundings, which is 30 watts. However, Cooper is smart and he covers the circular entrance with a radius $r = 0.2\,\mathrm{m}$ with a layer of large leaves that has a thickness of $0.98\,\mathrm{cm}$ and a thermal conductivity coefficient of $0.039\,\mathrm{W{\cdot}m^{-1}{\cdot}K^{-1}}$. This helps because heat does not escape from the burrow except through the entrance. Thus, Cooper can maintain a constant temperature of $20\,\mathrm{^\circ C}$ in the burrow even though it is $0\,\mathrm{^\circ C}$ outside. One night, a strong wind swept away all the leaves from the enclosure, leaving Cooper feeling cold once again.

Cooper opted to invite a friend into his burrow to sustain a cozy temperature of $20\,\mathrm{^\circ C}$. To achieve this, how many times heavier than Cooper must his friend be, considering our assumption that the heat output of wombats is directly proportional to their mass?

Káťa was excited that the Prague Zoo would finally have wombats.

When the entrance to the burrow is covered with leaves, the heat loss through the layer of leaves balances the wombat Cooper's heat output. If we consider the heat flux through the leaf layer to be homogeneous and steady, the relationship

$$\frac{P_\mathrm{C}}{S} = \lambda \frac{T_1 - T_2}{d}\,,$$

where P_C is the Cooper's heat capacity, $S = \pi r^2$ is the entrance area, λ is the thermal conductivity coefficient, d is the thickness of the leaf layer, and T_1, T_2 are the temperatures inside and outside the burrow respectively. We express Cooper's heat power and quantify

$$P_\mathrm{C} = \frac{\lambda(T_1 - T_2)\pi r^2}{d} \doteq 10\,\mathrm{W}\,.$$

The total heat loss of the burrow when the entrance is open is $P = 30\,\mathrm{W}$. Cooper's friend must, therefore, cover the heat output

$$P_\mathrm{k} = P - P_\mathrm{C} = 20\,\mathrm{W}\,.$$

Given our assumption that a wombat's heat output is directly proportional to its mass, Cooper requires a burrow companion with twice his heat output. Thus, Cooper's friend must possess double his mass.

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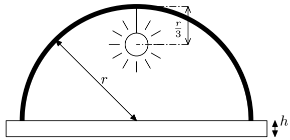
Problem DD ... street light



One can imagine a street light as a hemisphere of radius $r = 30\,\mathrm{cm}$*, in which the bulb hangs at a distance* $r/3$ *from the shell. The hemisphere is elevated above the ground at a height* $H = 4\,\mathrm{m}$ *from its center and is shielded from below by a glass lid of thickness* $h = 1\,\mathrm{cm}$*. By how many centimeters is the radius of the illuminated area reduced due to the hooded lamp, compared to the radius of the area it would illuminate if it were not hooded? The refractive index of the glass is* 1.5*, and you can neglect the scattering of light in the glass. Also, assume that the glass lid protrudes sufficiently over the edges of the lamp.*

David was thoroughly examining a street light on his way from a meeting.

To determine the size of the illuminated area, we are interested in the rays incident on the edge of the glass lid. Let us begin by calculating the horizontal distance that such a ray travels. In a lamp, it first travels a distance $R_{1,1} = r$ before it hits the glass lid. From simple geometry, we determine the sine of the angle of incidence of that ray as

$$\sin\alpha_1 = \frac{3\sqrt{13}}{13}\,.$$

After refraction, we get

$$\sin\alpha_2 = \frac{n_1}{n_2}\sin\alpha_1 = \frac{2\sqrt{13}}{13}\,,$$

from Snell's law. The ray travels a vertical distance h in the glass, so we can easily determine the horizontal distance as

$$R_{1,2} = h\cdot\tan\alpha_2\,.$$

It is essential to realize that the refracted ray will travel again at an angle α_1 from the normal. This time, it travels a vertical distance $H - h$, so similarly

$$R_{1,3} = (H-h)\cdot\tan\alpha_1\,.$$

The radius of the illuminated part is, therefore, in the hooded case

$$R_1 = R_{1,1} + R_{1,2} + R_{1,3} = \frac{3\,775}{6}\,\mathrm{cm}\,.$$

Non-hooded situation is simple – we can determine the radius of the illuminated area from the similarity of the triangles as

$$\frac{R_2}{r} = \frac{H + 2/3r}{2/3r} \quad\Rightarrow\quad R_2 = \frac{3H+2r}{2} = 630\,\mathrm{cm}\,.$$

Hence, we get that the radius of the area illuminated by the lamp without the lid is greater by

$$R_2 - R_1 = \frac{5}{6}\,\mathrm{cm} \doteq 0.833\,\mathrm{cm}\,.$$

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Problem DE ... honking trains

Danka is cycling along a straight train track at a speed v_D. In the same direction as her, a train goes down the railway at a speed $v_1 = 156\,\mathrm{km\cdot h^{-1}}$, honking its horn at a frequency of $f_1 = 330\,\mathrm{Hz}$. A second train comes from the opposite direction at a speed $v_2 = 65\,\mathrm{km\cdot h^{-1}}$ which is honking at $f_2 = 350\,\mathrm{Hz}$. However, Danka hears both trains making the same tone. At what speed is she traveling? Give the result in units of $\mathrm{km\cdot h^{-1}}$. *Danka went for a bike ride.*

Danka perceives a tone with a frequency different from the actual frequency produced by the train due to the train's motion relative to her. This phenomenon is rooted in the Doppler effect, where the crucial factor is the motion relative to the medium transporting the waves – in this scenario, the air serves as the medium for sound propagation. Both the trains as sources of sound and Danka as the receiver of sound are moving relative to the air, which, in our case, we consider to be stationary. In the case of those trains, the frequency heard by Danka is higher than that emitted by the trains because they are traveling towards Danka. In this case, for the frequency f that Danka hears from the first train, the formula looks like

$$f = f_1 \frac{c - v_\mathrm{D}}{c - v_1}\,,$$

where $c = 343\,\mathrm{m\cdot s^{-1}}$ is the speed of sound in air. In the fraction's numerator, Danka's speed is subtracted from the speed of sound because Danka is moving in the direction away from the train. Conversely, in the fraction's denominator, the train's speed is subtracted from the speed of sound as the train is moving toward Danka. She hears the same frequency f from the other train. In this case

$$f = f_2 \frac{c + v_\mathrm{D}}{c - v_2}\,,$$

where the sum in the fraction's numerator represents Danka's movement towards the other train, and the difference in the denominator represents the train's movement towards Danka. Now, we need to make these two equations equal and express the speed of Danka's motion

$$\begin{aligned}
f_1 \frac{c - v_\mathrm{D}}{c - v_1} &= f_2 \frac{c + v_\mathrm{D}}{c - v_2}\,,\\
(c - v_\mathrm{D})(c - v_2) f_1 &= (c + v_\mathrm{D})(c - v_1) f_2\,,\\
c(c - v_2) f_1 - v_\mathrm{D}(c - v_2) f_1 &= c(c - v_1) f_2 + v_\mathrm{D}(c - v_1) f_2\,,\\
c(c - v_2) f_1 - c(c - v_1) f_2 &= v_\mathrm{D}(c - v_2) f_1 + v_\mathrm{D}(c - v_1) f_2\,,\\
v_\mathrm{D} &= c \frac{(c - v_2) f_1 - (c - v_1) f_2}{(c - v_2) f_1 + (c - v_1) f_2}\,.
\end{aligned}$$

After plugging in the numerical values, we get

$$v_\mathrm{D} \doteq 3.80\,\mathrm{m\cdot s^{-1}} \doteq 13.7\,\mathrm{km\cdot h^{-1}}\,.$$

Danka is cycling at the speed $13.7\,\mathrm{km\cdot h^{-1}}$.

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Problem DF ... insomniac civlization

Imagine that an alien civilization has evolved on Venus and is floating on dense clouds of sulfuric acid at $h = 75\,\mathrm{km}$ above the surface. In order to survive, they need a constant supply of sunlight. The inhabitants must therefore live a nomadic life, always sailing towards the sunlight. What is the minimum average speed they must travel, if they live at 60 degrees latitude? The radius of Venus is 0.95 of the radius of the Earth, one rotation around its axis takes Venus 243 days, the orbit around the Sun takes 224 days, and its axis of rotation is approximately perpendicular to the orbital plane. Consider the circular orbit of Venus around the Sun and remember that Venus rotates in the opposite direction to its orbit around the Sun. *Matěj is scared of darkness.*

First, we calculate how long a day lasts when viewed from Venus, that is, how long it takes from noon to the following noon. This day is called a synodic day and we will denote it as T'. Since Venus rotates in the opposite direction than it orbits around the Sun, the synodic day is shorter than the rotation time around its axis T. Specifically, after one orbit of Venus around the Sun – after one period P – Venus rotates around axis $N = P/T$ times. However, in that time, $N + 1 = P/T'$ synodic days will pass, because one day is "hidden" in Venus' own orbit around the Sun. Combining these two equations, we get

$$\frac{P}{T} = \frac{P}{T'} - 1\,.$$

From this we express the length of the synodic day

$$T' = \frac{PT}{P+T} \doteq 117\,\mathrm{d}\,.$$

An alien civilization must sail around the planet in time T'. In order to to get their velocity v, we need to calculate the distance s which they will "swimm" through. The circumference of a parallel line is scaled with the cosine of the latitude φ. Moreover, we must not forget that the aliens are at the height $h = 75\,\mathrm{km}$ above the surface. This is at a distance $0.95R_\mathrm{Z} + h$ from the center of Venus, where $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ is the radius of the Earth. Thus

$$v = \frac{s}{T'} = 2\pi(0.95R_\mathrm{Z} + h)\frac{\cos}{\varphi}\frac{P+T}{PT} \doteq 1.9\,\mathrm{m{\cdot}s^{-1}} \doteq 6.9\,\mathrm{km{\cdot}h^{-1}}\,,$$

which is equivalent to brisk walking.

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Problem DG ... fast muons

How many times farther, on average, will a muon flying at $v_2 = 0.99c$ travel compared to a muon flying at $v_1 = 0.95c$? The mean lifetime of a muon is $\tau = 2.2\cdot10^{-6}\,\mathrm{s}$.
Danka recalled a lecture from special relativity.

Let us denote by t_1 the time of flight of the muon at velocity v_1 and by t_2 the time of flight at velocity v_2. Then, for the path the muon travels in each case, $x_1 = v_1t_1$ and $x_2 = v_2t_2$, respectively. The time of flight of a muon is given, among other things, by its mean lifetime. The crucial aspect is that the muon, as an unstable particle, only exists for a certain amount of time and then decays. Since the muon is moving at relativistic velocity, an observer on the ground, due to time dilation, will measure a longer lifetime for the muon than the time that the muon itself perceives. For time dilation, the relation is

$$t = \tau\gamma\,.$$

Here γ is the gamma factor, and it is equal to

$$\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}\,,$$

where v is the velocity of the muon relative to the observer on the ground, and c is the speed of light. Then, the ratio of the distances of the two muons can be expressed as

$$\frac{x_2}{x_1} = \frac{v_2 t_2}{v_1 t_1}\,,$$

$$\frac{x_2}{x_1} = \frac{v_2\tau}{v_1\tau}\frac{\sqrt{\frac{1}{1-\frac{v_2^2}{c^2}}}}{\sqrt{\frac{1}{1-\frac{v_1^2}{c^2}}}}\,,$$

$$\frac{x_2}{x_1} = \frac{v_2}{v_1}\sqrt{\frac{1-\frac{v_1^2}{c^2}}{1-\frac{v_2^2}{c^2}}}\,.$$

After substituting the given values, we get

$$\frac{x_2}{x_1} \doteq 2,3\,.$$

At a velocity of $0.99c$, a muon travels 2.3 times farther than at a velocity of $0.95c$.

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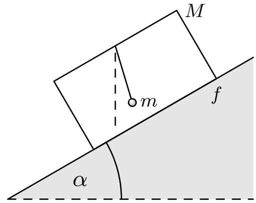
Problem DH ... cart with a plumb and friction

Consider a hill with a slope $\alpha = 30^\circ$. At the top, we put a hollow block of mass $M = 10\,\mathrm{kg}$ with a string of length $l = 15\,\mathrm{cm}$ hanging from the upper face and a point mass $m = 2.5\,\mathrm{kg}$ at the end of the string (this mass is not included in the the mass of the block). We release the block down the slope. At which angle (with respect to the vertical direction) does the string settle? The result should be positive if the string is tilted in the direction of travel, and negative if it is tilted in the opposite direction. The coefficient of friction between the block and the hill is $f = 0.10$.



Lego loves to iterate his problems.

We are interested in the steady state situation. In such a situation, the string and the point mass do not move relative to the block, as if they formed one perfectly rigid body together with the block. We can calculate the acceleration of this body down the hill.

Its total mass is $M+m$; the component of the gravitational force in the direction parallel to the hill is $(M+m)g\sin\alpha$. The component in the direction perpendicular to the hill (i.e. normal force) is $(M+m)g\cos\alpha$ and the force of friction (acting parallel to the hill and against the direction of motion) is thus $f(M+m)g\cos\alpha$. In total, the block is being accelerated by the force

$$F = F_\mathrm{p} - F_\mathrm{t} = (M+m)g\sin\alpha - f(M+m)g\cos\alpha,$$

and its acceleration is therefore

$$a = \frac{F}{M+m} = g(\sin\alpha - f\cos\alpha)\,.$$

More precisely, this is true as long as the friction is small enough to make the block actually slide down the hill. Otherwise, the acceleration would simply be 0. However, we can verify by substituting the values from the problem that the block indeed slides, since we get $a = 4\,\mathrm{m{\cdot}s^{-2}}$

Let's now move to the frame of reference accelerating along with the block. In order to make it so the point mass hanging from the rope in this system does not move, it must be subjected to a zero effective (total) force. Let's discuss the forces acting on it. Gravity mg acts vertically downwards; the inertial force ma acts parallel to the hill towards the rear of the block; and finally there is the force exerted by the rope on which the point mass hangs. The magnitude and direction of the force from the rope must be (in the steady state situation) exactly such that this force compensates for the resultant of the two remaining forces. It is important that the direction of the force from the rope is the same as the direction of the rope. Thus, we must find the direction of the resultant of the remaining two forces.

The force of gravity mg acts downwards. The vertical component of the inertial force has magnitude $ma\sin\alpha = mg(\sin\alpha - f\cos\alpha)\sin\alpha$ and it is directed upwards; the horizontal component has size $ma\cos\alpha$ and is directed backwards. The resultant of the forces of gravity and inertia thus has a vertical component of magnitude

$$F_\mathrm{vert} = mg\left(1 - \sin^2\alpha + f\cos\alpha\sin\alpha\right) = mg(\cos\alpha + f\sin\alpha)\cos\alpha$$

directed downwards and a horizontal component with magnitude

$$F_\mathrm{horiz} = mg(\sin\alpha - f\cos\alpha)\cos\alpha$$

directed backwards. Note that for the limiting case of a vertical hill ($\alpha = \pi/2$), the point mass in the reference frame of the block is not affected by any forces. This is due to the fact that this system is falling with acceleration g, and the point of mass is in a weightless state from the point of view in this reference frame.

However, let us return to the tilting angle of the rope. We are interested in the tilt with respect to the vertical direction, so we get this angle as the arctangent of the ratio of the horizontal component of the force to the vertical component

$$\beta = \arctan\frac{F_\mathrm{horiz}}{F_\mathrm{vert}} = \arctan\frac{\sin\alpha - f\cos\alpha}{\cos\alpha + f\sin\alpha}\,.$$

The string is tilted by the angle β towards the back, so the answer should be -24°.

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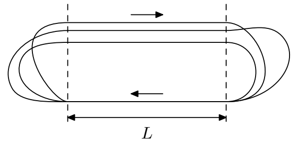
Problem EA ... mass points races

Lego is preparing for a competition of theoretical physicists, in which they compete against each other in a race of mass points. The mass points circle the track as follows. First, they travel along a straight line of length L, then they turn arbitrarily by 180°, and again travel along a straight line of length L, followed by another turn by 180°, and so on. Lego created his mass point such that it can reach the highest acceleration a, and its velocity relative to the track always has the same magnitude. Advise Lego on how to choose the magnitude of this velocity so that his mass point circles the track in the shortest possible time.



Lego is too clumsy to race with anything real.

The track consists of two identical straightaways and two identical curves, so we just need to keep track of the sum of the times the mass point takes to travel one straightaway of length L and one curve. For the velocity magnitude v, Lego's mass point traverses the straightaway in time $t_1 = L/v$.

Furthermore, for this velocity magnitude v, it will have to follow a circular path of radius for which applies

$$a = \frac{v^2}{R} \quad \Rightarrow \quad R = \frac{v^2}{a} \,.$$

The semicircular curve will have a length of $o/2 = \pi R$, therefore Lego's mass point will traverse it in time

$$t_2 = \frac{o/2}{v} = \frac{\pi R}{v} = \frac{\pi v}{a} \,.$$

We search to minimize the time

$$T = t_1 + t_2 = \frac{L}{v} + \frac{\pi v}{a} \,,$$

where the parameter over which we wish to minimize is v. That is, we differentiate T with respect to v, and we set the derivative equal to 0:

$$0 = \frac{\mathrm{d}T}{\mathrm{d}v} = -\frac{L}{v^2} + \frac{\pi}{a} \,,$$

$$\sqrt{\frac{La}{\pi}} = v \,,$$

and hence, we obtained the optimal speed.

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Problem EB ... a pasta problem

Adam noticed an interesting phenomenon at the dorms. The maximum possible water flow rate from a tap depends on its temperature. We know that the possible range of temperatures is $\langle t_1, t_2 \rangle$, water with temperature $t_1 = 20\,^\circ\mathrm{C}$ flows out with flow rate $Q_1 = 55\,\mathrm{ml{\cdot}s^{-1}}$, water with temperature $t_2 = 35\,^\circ\mathrm{C}$ flows out with a flow rate of $Q_2 = 400\,\mathrm{ml{\cdot}s^{-1}}$ and the relation between flow rate and temperature is linear. Adam needed to pour hot water from a pot of *pasta into the sink under the tap, so he started pouring hot water at a temperature of* $T = 80\,^\circ\mathrm{C}$ *with a flow rate of* $Q = 35\,\mathrm{ml{\cdot}s^{-1}}$. *Calculate the temperature of the water that has to come out of the tap to allow the temperature of the water flowing out of the sink to be as low as possible. Consider that the drain of the sink is large enough that all the water flows out of the sink quickly without pooling up, all the water in the sink mixes quickly to an equilibrium temperature, and neglect heat loss to the surroundings. Assume that the density of water is constant.* *Adam doesn't want to torment the pipes.*

Let's start by writing the calorimetric equation for a simpler situation – we have two constant amounts of water at two initial temperatures in the sink.

$$m \cdot c \cdot (t_\mathrm{v} - t) = M \cdot c \cdot (T - t_\mathrm{v})\,,$$

where c represents the specific heat capacity of water, m represents the mass of water that we let out of the tap, t its temperature, M the mass of water that Adam pours into the sink, and t_v is the resulting temperature which we are trying to minimize. This notation is indeed not very correct in our situation, because instead of masses, we have flow rates – but note that if we divide both sides of the equation by the specific heat capacity, then we divide both sides by the density of water and then once again by time, we get a notation that we can already apply to our situation. Specifically,

$$q \cdot (t_\mathrm{v} - t) = Q \cdot (T - t_\mathrm{v}) \quad \Rightarrow \quad t_\mathrm{v} = \frac{QT + qt}{Q + q}\,. \tag{3}$$

Now to determine how the flow rate of the water flowing out of the tap depends on its temperature. From the problem statement, we know that the dependence is linear, so we look for a dependence in the form

$$q = a \cdot t + b\,.$$

By substituting the values in the problem and solving the system of equations, we easily get

$$q = (23\,\mathrm{ml{\cdot}s^{-1}{\cdot}K^{-1}}) \cdot t - (405\,\mathrm{ml{\cdot}s^{-1}})\,.$$

This relationship is plugged into the equation (3). For simplicity, we shall consider everything to be dimensionless.

$$[t_\mathrm{v}] = \frac{23[t]^2 - 405[t] + 2800}{23[t] - 370}\,.$$

We now differentiate this expression by the dimensionless temperature $[t]$ and we set this derivative to be equal to zero. After simplifying, we get

$$\frac{529[t]^2 - 17020[t] + 85450}{(23[t] - 370)^2} = 0\,.$$

It is enough if the numerator of the fraction is equal to zero. Solving this quadratic equation, we get only one solution satisfying the problem, namely that the minimum we are looking for lays at $[t] \doteq 25.95$, so the answer to the original question is roughly $26\,^\circ\mathrm{C}$.

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Problem EC ... Another bead on a string

Consider a bead of mass m strung on a perfectly rigid circular wire of radius r, which is fixed in space. We take a spring of stiffness k with zero proper length, attach one of its endpoints to a fixed point at a distance $r/2$ from the center of the circle and the other endpoint to the bead. What is the period of the small oscillations of the bead around its equilibrium position?

Lego has noticed that he has not made enough problems about oscillations this year.

The equilibrium position of the bead is, of course, when it is closest to the point where the other end of the spring is attached, since the length of the spring in that position is $l_0 = r/2$. The question is what changes when the bead moves by a distance Δo along the circle. We can find this more easily by introducing a coordinate system with the origin at the center of the circle along which the bead can move. Let's choose the rotation of its axes so that in the equilibrium position, the bead has coordinates $[r, 0]$ (and then the fixed endpoint of the spring has coordinates $[r/2, 0]$).

Solution via energy

When the bead moves by Δo along the circumference of the circle, it moves by an angle $\varphi = \Delta o/r$ with respect to its center, where $\varphi \ll 1$. Then we can simply write the new coordinates of the bead as $[r\cos\varphi, r\sin\varphi]$. The length of the spring in this position is obtained by calculating the distance of this point from the fixed endpoint of the spring, which we can do using Pythagoras' theorem. The difference of the x-coordinates is $r(\cos\varphi - 1/2)$ and the difference of the y-coordinates is $r\sin\varphi$. Then the length of the spring is

$$l = \sqrt{r^2\left(\cos\varphi - \frac{1}{2}\right)^2 + r^2\sin^2\varphi} = r\sqrt{\cos^2\varphi - \cos\varphi + \frac{1}{4} + \sin^2\varphi} = r\sqrt{1 - \cos\varphi + \frac{1}{4}}\,,$$

while for small angles φ, the approximation $\cos\varphi \approx 1 - \varphi^2/2$ holds. We should further note that we are not so much interested in the length of the spring itself as the change in its energy with respect to the equilibrium position. Since the energy of the spring is $kl^2/2$, the increase in energy from the equilibrium position is $k(l^2 - l_0^2)/2$, and we approximate

$$l^2 - l_0^2 \approx r^2\left(\frac{\varphi^2}{2} + \frac{1}{4}\right) - \frac{r^2}{4} = \frac{r^2}{2}\varphi^2\,.$$

We substitute $\varphi = \Delta o/r$ and express the change of the spring's energy

$$\Delta E_\mathrm{p} = \frac{1}{2}k(l^2 - l_0^2) = \frac{1}{2}\frac{k}{2}\Delta o^2\,.$$

We are interested in the stiffness the bead feels when it oscillates. We can find it simply by treating the change of the spring's energy as $k_\mathrm{eff}\Delta o^2/2$, where k_eff is the stiffness of a spring that would oscillate in the same way when attached to the bead with no additional constraints on movement. Comparing the expressions, we get $k_\mathrm{eff} = k/2$.

Solution via force

The second option is to calculate the force on the bead after displacement by Δo. Only the spring and wire are acting on the bead, while the force from the wire just cancels out the

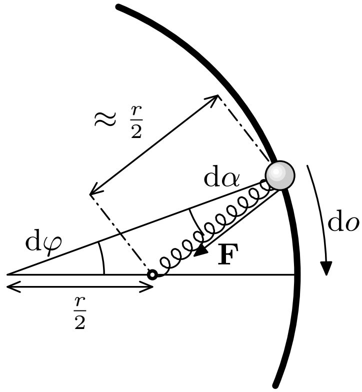

Fig. 2: Depiction of the situation.

component of the force from the spring which is perpendicular to the wire (because the bead cannot move in that direction). The resulting force on the bead is the component of the force from the spring which is parallel to the direction of the wire at the bead's current position.

For small oscillations, we can neglect the change in the spring's length and thus assume that the magnitude of the force exerted by the spring is $F_0 = kr/2$. We just need to find the projection of this force in the direction parallel to the wire. This projection is equal to $\Delta F = F_0 \sin\alpha$, where α is the angle between the direction in which the spring is pulling and the direction perpendicular to the wire. The spring is pulling the bead directly towards the fixed endpoint. Since the wire is circular, the direction perpendicular to the wire at any point is always towards the center of the circle.

When we draw the situation, we see that the triangle (with vertices: center of the circle, fixed attachment point of the spring, position of the bead) is, for a sufficiently small displacement, an approximately isosceles triangle. The angles opposite to its legs are congruent, and therefore $\alpha \approx \varphi \ll 1$. It remains to take advantage of the approximation $\sin\varphi \approx \varphi$, substitute $\varphi = \Delta o/r$, and we find out that the force on the bead after it moves by Δo from the equilibrium position is

$$\Delta F = F_0 \sin\Delta\alpha \approx k\frac{r}{2}\varphi = \frac{k}{2}\Delta o\,.$$

The stiffness that the bead "feels" satisfies $\Delta F = k_{\text{eff}}\Delta o$. By comparing the expressions for the force, we get that the bead is being pulled back to the equilibrium position as if by a spring with an effective stiffness $k_{\text{eff}} = k/2$.

It seems almost miraculous, but these two completely different approaches actually give the same result.

Result

Whether using energies or forces, we have concluded that the bead oscillates as if it were attached to a spring with a stiffness of $k_{\text{eff}} = k/2$. Finally, we note that as far as kinetic energy or inertia is concerned, the curvature of the trajectory of the bead may be neglected. This means that we can use the known formula for the period of small oscillations, only substituting k_{eff}, to obtain

$$T = 2\pi\sqrt{\frac{m}{k_{\mathrm{eff}}}} = 2\pi\sqrt{2\frac{m}{k}}\,.$$

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Problem ED ... let's increase intensity

In many engineering applications, we need to obtain high intensities of electric field. We use a wire with a diameter of $3.0\,\mathrm{mm}$ *as one electrode, which we insert on the axis of a second, hollow cylindrical electrode. Its radius is* $1.2\,\mathrm{cm}$*. We apply a voltage* U *to the electrodes. We compare this circuit with a plate capacitor where we apply the same voltage, and the distance between the plates is* $2.0\,\mathrm{cm}$*. Determine the ratio of the electrical intensity in the close surroundings of the wire in the first circuit to the intensity in the plate capacitor in the second circuit.*

Jarda wanted to observe St. Elmo's fire.

Let us denote the radius of the large cylindrical electrode $b = 1.2\,\mathrm{cm}$, the radius of the small one as $a = 1.5\,\mathrm{mm}$, and the distance between the plates of the capacitor $d = 2.0\,\mathrm{cm}$. The intensity of electric field in a plate capacitor is simply

$$E_{\mathrm{d}} = \frac{U}{d}\,.$$

For a cylindrical capacitor the situation is more complicated. Because the narrower of the electrodes is cylindrical, it creates a field around it that decreases in proportion to r^{-1}, where r is the distance from the axis of symmetry. There is a potential difference U between the electrodes, which can be written in terms of

$$U = \int_a^b E(r)\,\mathrm{d}r = \int_a^b \frac{c}{r}\,\mathrm{d}r = c\ln\left(\frac{b}{a}\right)\,.$$

From here, we express the proportionality constant c and find the electric field around the electrode, i.e.

$$E_a = \frac{c}{a} = \frac{U}{a\ln\left(\frac{b}{a}\right)}\,.$$

The wanted ratio is thus

$$\frac{E_a}{E_{\mathrm{d}}} = \frac{d}{a\ln\left(\frac{b}{a}\right)} = 6.4\,.$$

By changing the geometry, we can achieve higher field strengths in some places, even with similar external dimensions.

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Problem EE ... Fykosaurus

The Fykosaurus usually flies over the landscape with a velocity $v_0 = 80\,\mathrm{km{\cdot}h^{-1}}$ in the horizontal direction. At this velocity, the force of dynamic lift is the same as the force of gravity. However, if the Fykosaurus wants to rest on a tree, he must land on it – to do so, he must fly with a velocity of at most $v_1 = 20\,\mathrm{km{\cdot}h^{-1}}$ in the horizontal direction. Suppose that he uniformly decelerates to this velocity in this direction with a deceleration of $a = 2\,\mathrm{m{\cdot}s^{-2}}$. What would his vertical velocity on landing be if he had zero vertical velocity before he started decelerating and was not accelerating upward?
Note: Dynamic lift is proportional to the square of horizontal velocity.

David was afraid of dropping out of Matfyz.

First, we calculate how long it would take the Fykosaurus to reach the landing velocity v_1. Since the deceleration is constant, the

$$v_1 = v_0 - aT \quad \Rightarrow \quad T = \frac{v_0 - v_1}{a}\,,$$

where v_0 is the initial velocity, a is the deceleration in the horizontal direction, and T is the sought time. Thanks to the condition for F_{dyn}, we obtain an equation that allows us to calculate the constant of proportionality

$$F_{\mathrm{dyn}} = k \cdot v_0^2 = mg\frac{mg}{v_0^2}\,.$$

From Newton's second law, the vertical acceleration a_{V} is

$$a_{\mathrm{V}} = \frac{F}{m} \quad \Rightarrow \quad a_{\mathrm{V}} = \frac{mg - F_{\mathrm{dyn}}}{m}\,.$$

Since we are interested in vertical velocity, we only need to integrate this acceleration with respect to time. If we denote the horizontal velocity of the Fykosaurus by v_{H}, we can write

$$\begin{aligned} v_{\mathrm{landing}} &= \int_0^T a_{\mathrm{V}}\,\mathrm{d}t = \int_0^T \frac{mg - F_{\mathrm{dyn}}}{m}\,\mathrm{d}t = \\ &= \int_0^T \frac{mg - \frac{mg}{v_0^2}\cdot v_{\mathrm{H}}^2}{m}\,\mathrm{d}t = \\ &= g\int_0^T 1 - \frac{1}{v_0^2}\cdot v_{\mathrm{H}}^2\,\mathrm{d}t \end{aligned}$$

Finally, we substitute $v_{\mathrm{H}} = v_0 - at$ and the resulting integral is easily calculated as

$$\begin{aligned} v_{\mathrm{landing}} &= g\int_0^T 1 - \frac{(v_0 - at)^2}{v_0^2}\,\mathrm{d}t = g\int_0^T \frac{2at}{v_0} - \frac{a^2t^2}{v_0^2}\,\mathrm{d}t = \\ &= g\frac{2a}{v_0}\left[\frac{t^2}{2}\right]_{t=0}^T - g\frac{a^2}{v_0^2}\left[\frac{t^3}{3}\right]_0^T = g\frac{-aT^2}{v_0} - g\frac{a^2T^3}{3v_0^2} = \\ &= g\frac{a}{v_0}\frac{(v_0 - v_1)^2}{a^2} - g\frac{a^2}{3v_0^2}\frac{(v_0 - v_1)^3}{a^3} = g\frac{3v_0\,(v_0 - v_1)^2 - (v_0 - v_1)^3}{3v_0^2 a} = \\ &= g\frac{(v_0 - v_1)^2(2v_0 + v_1)}{3v_0^2 a}\,. \end{aligned}$$

After substitution, we get $v_{\text{landing}} \doteq 46\,\mathrm{m\cdot s^{-1}}$. We see that the birds in general must also be slowing down significantly in the vertical direction during landing.

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Problem EF ... imperfect diode

Consider a diode that has a resistance $2.2\,\Omega$ *in the forward direction, while in the reverse direction, its resistance is infinite. The imperfection of the diode lies in its non-zero parasitic capacitance of* $19\,\mathrm{pF}$*, which can be represented as a capacitor connected in parallel to the diode's resistance in an equivalent circuit. We connect such a diode to an AC voltage source with a frequency of* $3.5\,\mathrm{GHz}$ *and a voltage amplitude of* $12\,\mathrm{V}$*. Determine the ratio of the maximum current in the forward direction to the maximum current in the reverse direction. You should give a positive value.* *Jarda was driving in the wrong direction on a one-way street.*

In parallel wiring of a capacitor and a resistor, we have the same voltage on both elements, which in our case is equal to the voltage on the source $U = U_0 \sin(\omega t)$, where $\omega = 2\pi f$ is the angular frequency of the source. Thus

$$U = RI_R\,,$$
$$U = \frac{Q}{C}\,,$$

where R is the resistance of the resistor and I_R is the current flowing through it. The charge on the capacitor of capacitance C is Q. The current through the branch with capacitance corresponds to

$$I_C = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}U}{\mathrm{d}t} = U_0 \omega C \cos(\omega t)\ .$$

The total current through the diode is determined as the sum of the currents through the two branches

$$I = U_0\left(\frac{1}{R}\sin(\omega t) + C\omega\cos(\omega t)\right)\ .$$

The capacitive current remains the same for both directions, only with the opposite sign, while the resistive current varies depending on the direction. Let the diode be connected in the forward direction during the first half of the period $T = 1/f$. Then, the current satisfies

$$I_{0<t<T/2} = U_0\left(\frac{1}{R_1}\sin(\omega t) + C\omega\cos(\omega t)\right)\ ,$$

with $R_1 = 2.2\,\Omega$. However, after substituting $R_2 = \infty$ in place of R_1, we obtain

$$I_{T>t>T/2} = U_0 C\omega\cos(\omega t)\ .$$

The maximum current in the forward direction is found by differentiating the expression $I_{0<t<T/2}$ with respect to time and setting it equal to zero

$$\frac{\mathrm{d}I_{0<t<T/2}}{\mathrm{d}t} = U_0\left(\frac{1}{R_1}\omega\cos(\omega t) - C\omega^2\sin(\omega t)\right) = 0$$
$$\Rightarrow \frac{1}{R_1 C\omega} = \tan(\omega t_m)$$
$$\Rightarrow I_{0<t<T/2,max} = \frac{U_0}{R_1}\sqrt{R_1^2C^2\omega^2 + 1}\,.$$

The value of the maximum current in the reverse direction is $I_{T>t>T/2,max} = U_0 C\omega$. The ratio we are looking for is then

$$\frac{I_{0<t<T/2,max}}{I_{T>t>T/2,max}} = \frac{\sqrt{R_1^2 C^2 \omega^2 + 1}}{R_1 C\omega} = \frac{\sqrt{4\pi^2 R_1^2 C^2 f^2 + 1}}{2\pi R_1 C f} = 1.48\,.$$

Here we can observe that for $C \to 0$, the ratio tends to infinity as we would expect. On the other hand, for high frequencies or large capacitances, the ratio tends to one, i.e. the asymmetric conductance is no longer observed. This must be taken into account when designing high-frequency circuits.

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Problem EG ... pressed hatch

Consider a space with an opening of area A, which we close with a straight vertical cylindrical hatch of the same inner and outer area A. We pump the air out of the space to a pressure of 10 Pa and then shut down the pump. There are, however, leaks between the hatch and the rest of the vacuum apparatus, so that air particles still enter from the surrounding atmosphere. Consider that the rate of leakage is directly proportional to the pressure difference between the compartments and inversely proportional to the force by which the hatch is pressed against the apparatus. After seven hours since the air was pumped out, a pressure of 80 Pa was measured inside. Determine how long after the apparatus has been pumped will the pressure exceed 200 Pa if the temperature inside the apparatus is $T = 20\,^\circ\mathrm{C}$ all the time.

Jarda's blood pressure is rising.

Let's rewrite the assignment as an equation for the number of particles inside the hatch N

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \alpha \frac{(p_\mathrm{a} - p_\mathrm{in})}{F}\,, \tag{4}$$

where $\mathrm{d}N/\,\mathrm{d}t$ is the change of the number of particles inside the apparatus in time, α is some yet unknown coefficient of proportionality, p_a is the atmospheric pressure outside the apparatus, p_in is the pressure inside which changes with time, and F is the force applied to the hatch by the rest of the apparatus.

Now let's express several other relationships. The number of particles inside is obviously

$$N = N_\mathrm{A} n\,,$$

where N_A is Avogadro's constant and n is the amount of substance of the gas inside. From the ideal gas law, we can derive the relation between n, V, and the pressure p_in as

$$p_\mathrm{in} V = nRT\,,$$

where T is the thermodynamic temperature. We can see that the number of particles inside the apparatus is directly proportional to the pressure p_in, since all the other variables are constant during the process

$$N = \frac{p_\mathrm{in} V}{RT} N_\mathrm{A}\,,$$

The force acting on the hatch from the outside is $F_{\mathrm{out}} = Ap_{\mathrm{a}}$. According to the problem, the area on the inside is the same as from the outside, so the force acting from the inside is $F_{\mathrm{in}} = Ap_{\mathrm{in}}$. Therefore, the total force acting on the hatch is $F = A\left(p_{\mathrm{a}} - p_{\mathrm{in}}\right)$. We can see that it is also proportional to the pressure difference between the interior and exterior. It's worth noting that, since the hatch is placed vertically, it is not pushed against the apparatus by its gravitational force.

Let us substitute N and F into our differential equation (4). We get

$$\frac{\mathrm{d}p_{\mathrm{in}}}{\mathrm{d}t} = \frac{\alpha RTA}{VN_{\mathrm{A}}}\frac{\left(p_{\mathrm{a}} - p_{\mathrm{in}}\right)}{\left(p_{\mathrm{a}} - p_{\mathrm{in}}\right)} = \beta\,,$$

where we denote the constant $\alpha RTA/(VN_{\mathrm{A}}) = \beta$ and truncate the dependence on the pressure difference. The pressure in the apparatus increases linearly as

$$p_{\mathrm{in}} = p_{\mathrm{in0}} + \beta t\,,$$

where t is the time from reaching the pressure, $p_{\mathrm{in0}} = 10\,\mathrm{Pa}$. We also know that after time $t_0 = 7\,\mathrm{h}$ the pressure increased to $p_{\mathrm{in1}} = 80\,\mathrm{Pa}$. From here, we can calculate the constant β as

$$p_{\mathrm{in1}} = p_{\mathrm{in0}} + \beta t_0 \quad \Rightarrow \quad \frac{p_{\mathrm{in1}} - p_{\mathrm{in0}}}{t_0} = \beta\,.$$

We reach the pressure $p_{\mathrm{in2}} = 200\,\mathrm{Pa}$ at time τ, which we find as

$$p_{\mathrm{in2}} = p_{\mathrm{in0}} + \beta\tau \quad \Rightarrow \quad \tau = \frac{p_{\mathrm{in2}} - p_{\mathrm{in0}}}{\beta} = t_0\frac{p_{\mathrm{in2}} - p_{\mathrm{in0}}}{p_{\mathrm{in1}} - p_{\mathrm{in0}}} = 19\,\mathrm{h}\,.$$

Pressure 200 Pa will be in the apparatus 19 h from the first depletion to 10 Pa.

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Problem EH ... asynchronous single-phase motor

One approach to construct an electric motor using a single-phase AC voltage source, involves connecting two branches in parallel. In one of these branches, the phase can be shifted by introducing a capacitor. We need the phase difference between the branches to be $\pi/2$ and the electric current amplitude in each branch to be the same. Each branch contains two coils connected in series with an inductance L and a resistance R. How do we need to choose the angular frequency of the source ω and the capacitance of the capacitor C, which will be connected in series with one branch, to achieve the desired outcome?

Lego had lecture about electric motors at the camp.

The condition for currents connected to one source to have the same amplitude is that both branches also have the same amplitude. Furthermore, if we require that the currents are offset by $\pi/2$, the angle between the vectors representing the impedances in the complex plane must also be equal to $\pi/2$.

The impedance of the branch without a capacitor will be

$$Z_1 = 2R + i2\omega L\,,$$

where i denotes the complex unit. The impedance of the branch with a capacitor will be

$$Z_1 = 2R + i\left(2\omega L - \frac{1}{\omega C}\right) .$$

We can see that the real component of the impedance is the same, and the condition for the impedance to be the same implies that the complex component must also be the same size. If the impedances are to be offset relative to each other, they must be complex conjugates $Z_1 = \bar{Z}_2$. In other words, the complex components must be opposite

$$2\omega L = -\left(2\omega L - \frac{1}{\omega C}\right) ,$$
$$4\omega L = \frac{1}{\omega C} .$$

If we also want the phase difference to be $\pi/2$, combined with $Z_1 = \bar{Z}_2$, we get that the angles between the two impedances and the real axis must be $\pi/4$. This means that the real and complex components must be equal

$$2\omega L = 2R \rightarrow \omega = \frac{R}{L} .$$

Thus, we found the necessary angular frequency. It remains to substitute it into the previous equation

$$4\frac{R}{L}L = \frac{1}{\frac{R}{L}C}$$
$$C = \frac{L}{4R^2} .$$

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Problem FA ... irresistibly attractive reloaded

Jindra finally found himself a girlfriend, and so he decided to sell his unneeded black hole from problem 8 in Physics Brawl Online 2023 to someone else. However, he noticed that his black hole is shrinking due to the Hawking's radiation. Jindra's black hole had initial mass of $3.675 \cdot 10^{12}$ kg. *The relation for temperature of Hawking's radiation is*

$$T = \frac{\hbar c^3}{8\pi G M k} ,$$

where M is the mass of the black hole and $\hbar, c, G, k$ are reduced Planck's constant, the speed of light in vacuum, gravitational constant, and Boltzmann constant. Furthermore, for the Schwarzschild radius of the black hole holds $R_\mathrm{s} = 2GM/c^2$. Assume that the black hole radiates only photons from the event horizon and does not gain any mass from its surroundings. In how many years will Jindra's black hole evaporate? *Jindra says hello to Denča in Frenštát.*

If we make the assumption that the black hole emits only photons from the event horizon, then it radiates with the power

$$P = 4\pi R_\mathrm{s}^2 \sigma T^4 \tag{5}$$

according to the Stefan-Boltzmann law, where σ is the Stefan-Boltzmann constant, T is the temperature of the black hole, and

$$R_\mathrm{s} = \frac{2GM}{c^2} \tag{6}$$

is the Schwarzschild radius of the black hole depending on the mass M of the black hole. As we know from the problem, the temperature of a black hole is also dependent on its mass

$$T = \frac{\hbar c^3}{8\pi GMk}\,. \tag{7}$$

The initial temperature of Jindra's black hole was $T_0 = 3.34\cdot 10^{10}\,\mathrm{K}$, which is about three orders of magnitude higher than the temperature within the Sun. The Stefan-Boltzmann constant may be expressed using other fundamental constants

$$\sigma = \frac{\pi^2 k^4}{60c^2\hbar^3}\,. \tag{8}$$

Now, we substitute the relations (6), (7), and (8) to the equation (5), and adjust

$$P = 4\pi\frac{4G^2M^2}{c^4}\frac{\pi^2k^4}{60c^2\hbar^3}\frac{\hbar^4c^{12}}{4096\pi^4G^4M^4k^4} = \frac{\hbar c^6}{15360\pi G^2M^2}\,.$$

We have derived the formula for the luminosity of a black hole depending just on its mass M. The luminostity derived under the assumptions of the problem (photons only, emission from the event horizon) is called the Bekenstein-Hawking luminosity.

Since the black hole radiates energy, and no matter is falling into it from the surroundings, according to the assumptions of the problem statement, it will lose mass. Eintein's relation $E = mc^2$ relates energy to mass. Thus, for the loss of mass of a black hole, we get the differential equation

$$-c^2\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\hbar c^6}{15360\pi G^2M^2}\,.$$

The initial mass of the black hole is $M_0 = 3.675\cdot 10^{12}\,\mathrm{kg}$ and its final mass after evaporation is zero. The initial time is $t = 0$ and the black hole evaporation time is T. Using these integration bounds, we can solve the differential equation

$$-\int_{M_0}^{0} M^2\,\mathrm{d}M = \frac{\hbar c^4}{15360\pi G^2}\int_0^T \mathrm{d}t$$

$$\frac{1}{3}M_0^3 = \frac{\hbar c^4}{15360\pi G^2}T$$

$$T = \frac{5120\pi G^2}{\hbar c^4}M_0^3 = 4.172\cdot 10^{21}\,\mathrm{s}.$$

Jindra's black hole will evaporate after $4.172\cdot 10^{21}\,\mathrm{s}$, which corresponds to $1.32\cdot 10^{14}$ years.

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Problem FB ... Rydberg state

What principal quantum number must an electron in a hydrogen atom have to be one astronomical unit away from the nucleus? Consider Bohr's model of the atom.

Jarda felt detached from reality.

To solve the problem, we will use Bohr's model of the hydrogen atom. Here, the electron is attracted to the nucleus electrostatically by the Coulomb force

$$F_\mathrm{C} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,,$$

where ε_0 is the vacuum permittivity, e is the charge of the electron and proton (elementary charge), and r represents the electron's distance from the nucleus of the atom.

This force is equal to the centripetal force

$$F_\mathrm{d} = m_\mathrm{e}\frac{v^2}{r}\,,$$

which acts on an electron of mass m_e orbiting at velocity v. An important postulate in Bohr's model is the quantization of the angular momentum of the electron

$$L = m_\mathrm{e}v\,r = n\hbar\,,$$

where n is the principal quantum number and $\hbar$ the reduced Planck constant. Alternatively, this condition can be expressed by determining the integer number of wavelengths for an electron on a circular trajectory with a length of $2\pi r$. Here, the wavelength is defined as $\lambda = h/p$, where p represents the momentum.

Substituting from the quantization condition for the electron velocity into the equation of forces, we get

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m_\mathrm{e}\frac{1}{r}\left(\frac{n\hbar}{m_\mathrm{e}r}\right)^2 \quad\Rightarrow\quad r = n^2\frac{4\pi\varepsilon_0\hbar^2}{m_\mathrm{e}e^2} = n^2\cdot 5.297\cdot 10^{-11}\,\mathrm{m}\,.$$

By substituting into the condition in the problem statement

$$1\,\mathrm{AU} = r = n^2\cdot 5.297\cdot 10^{-11}\,\mathrm{m} \quad\Rightarrow\quad n = \sqrt{\frac{1\,\mathrm{AU}}{5.297\cdot 10^{-11}\,\mathrm{m}}} \doteq 53\cdot 10^9\,.$$

Regarding highly excited electrons in atoms, we speak of Rydberg states. Scientists have observed electrons hundreds of nanometres away from the nuclei, which is extremely large by microworld standards. For electrons that are far away, only a tiny amount of energy is necessary for ionization. Our scenario involves distances spanning many orders of magnitude greater, so we could not replicate such behavior under terrestrial conditions.

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Problem FC ... resistance of a reaction

Consider a circuit with a DC voltage source with voltage $U = 450\,\mathrm{mV}$*, which is connected to an electrolyte using electrodes. For charge to flow through the circuit, a certain resistance must be overcome as the charge passes from the electrolyte to an electrode and vice versa. Consider an alternative circuit with a resistor* R_p*, which represents the transition between the electrolyte and the electrodes, and a resistor with resistance* $R_\mathrm{o} = 28\,\mathrm{m\Omega}$*, which represents the ohmic losses in the whole circuit, connected in series. However, the resistance* R_p *depends on the voltage* u *across it as* $R_\mathrm{p} = R_\mathrm{p0}\exp(-u/\alpha)$*, where* $\alpha = 100\,\mathrm{mV}$ *and* $R_\mathrm{p0} = 7.0\,\Omega$*. Determine the current that flows through the circuit.* *Jarda is still processing data from his bachelor's thesis.*

The current flowing through the circuit can be expressed as

$$I = \frac{U}{R_\mathrm{p} + R_\mathrm{o}}\,,$$

but R_p is dependent on the current.

The voltage on the resistor R_p is

$$u = U - IR_\mathrm{o}\,.$$

By modifying the first equation and substituting for R_p, we get

$$U - IR_\mathrm{o} = IR_\mathrm{p} = IR_\mathrm{p0}\exp\left(-\frac{u}{\alpha}\right) = IR_\mathrm{p0}\exp\left(-\frac{U - IR_\mathrm{o}}{\alpha}\right)\,.$$

With further adjustments, we modify it to the form

$$u = (U - u)\,\frac{R_\mathrm{p0}}{R_\mathrm{o}}\exp\left(-\frac{u}{\alpha}\right)\,,$$

from which we get

$$\frac{x}{(\xi - x)} = \beta\exp(-x)\,,$$

where we have established dimensionless variables $x = u/\alpha$, the ratio $\beta = R_\mathrm{p0}/R_\mathrm{o} = 250$ and the ratio $\xi = U/\alpha = 4.5$. We did all this to solve an equation that has no analytical solution.

We apply the natural logarithm to both sides of the equation and get

$$x = \ln\left(\frac{\beta\,(\xi - x)}{x}\right) = \ln\left(\frac{250\,(4.5 - x)}{x}\right)\,.$$

Let's first try to guess an approximate solution. If we put $x = 2$ on the right hand side, then $x = 5.745$ on the left hand side. We plug this value back into the right hand side, but get a negative argument of the logarithm, which is definitely not correct. However, we can see that we probably hit the value of x at least within an order of magnitude.

Let's try putting $x = 3$ on the right hand side. We get $x = 4.828$, which leads to the same problem as before. For $x = 4$ we get 3.442 from the right hand side function, so the correct value should lie somewhere in between. By sequentially halving the interval and plugging in the values $3.5, 3.75, 3.87, 3.81$, we arrive at the value 3.813. Trying 3.811, we get $x = 3.811\,06$, which is already a very good match. We have quickly found the numerical solution to the equation and can express the current I as

$$I = \frac{U - u}{R_\mathrm{o}} = \frac{U - x\alpha}{R_\mathrm{o}} = 2.46\,\mathrm{A}\,.$$

A current of 2.46 A flows through the circuit.

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Problem FD ... flag waving in the wind

Consider a flag as a rigid homogeneous rectangle with mass m, a horizontal side of the length a, and a vertical side of the length b, which can freely rotate around one of its vertical sides.

Wind is blowing with the velocity v. Assume that the total interaction between the flag and air is described by a single force with magnitude $F = KSv^2$, where S is the area of projection of the flag onto a plane perpendicular to the wind direction, and K is a constant. The force is acting in the direction perpendicular to the flag uniformly along its whole surface.

Find the period of small oscillations of the flag.

Legolas wanted to make an approximation of a flag.

When the flag is tilted by a small angle $\varphi \ll 1\,\mathrm{rad}$ into the wind, the area of its projection onto a plane perpendicular to this direction is $S = ab\sin\varphi \approx ab\varphi$. The magnitude of the force acting on it is $F = Kv^2ab\varphi$.

What is the torque exerted on the flag by this force? The problem statement says that the force is acting uniformly along the whole surface of the flag, so the "center of mass" of this force is in the middle of the flag. It also says that this force is perpendicular to the flag, so its moment arm is $a/2$. We find out that the torque acting against the direction of displacement (tilt), when the flag is tilted by an angle φ, is $M = Fa/2 = Kv^2a^2b\varphi/2$. Therefore, the torsion constant (a kind of "angular stiffness") is $D = Kv^2a^2b/2$.

The only other property we need to find is the moment of inertia of the flag. We can ignore the direction in which it cannot turn, so our task is to find the moment of inertia of a rod with mass m and length a, around an axis passing through its endpoint, which is $I = ma^2/3$.

The remaining question is how to find the period of oscillations from these intermediate values. Either we know / find (in physics tables) the formula for the period of a physical pendulum, which is

$$T = 2\pi\sqrt{\frac{I}{D}}\,,$$

where I is the moment of inertia of the pendulum with respect to the axis of rotation and D is the torsion constant with respect to the same axis. For a typical physical pendulum, $D = mga$, where m is the mass, g is the acceleration due to gravity, and finally a is the distance of the axis of rotation from the center of mass. In our case, of course, D is something completely different, since the force which causes the oscillations is not gravity, but drag force of wind. The meaning of torsion constant as "angular stiffness" remains the same, however. The formula $T = 2\pi\sqrt{I/D}$ may also be guessed using dimensional analysis as an analogy to the formula for the period of a linear harmonic oscillator $T = 2\pi\sqrt{m/k}$.

If the motivation from the previous paragraph is sufficient for you, you may skip this paragraph. Otherwise, the full derivation comes simply from the Newton's second law for rotation

$$I\ddot{\varphi} = M\,,$$

where M is the torque acting on the body. When we substitute that the torque which returns the flag to its equilibrium position is $M = -D\varphi$, with D as our torsion constant and φ as the angular displacement of the flag from the equilibrium position, we obtain a differential equation of the second order

$$I\ddot{\varphi} = -D\varphi\,,$$

for which the solution is (feel free to verify it by substituting back)

$$\varphi(t) = \varphi_0 \sin\left(\sqrt{\frac{D}{I}}t + \psi_0\right)\,,$$

where ψ_0 and φ_0 are constants determined from initial conditions. The important part is that the period of this motion is the (smallest) time which, when added to t, does not change the phase (i.e. changes the phase by 2π), which gives the equation

$$\sqrt{\frac{D}{I}}t + \psi_0 + 2\pi = \sqrt{\frac{D}{I}}(t+T) + \psi_0\,,$$

$$2\pi = \sqrt{\frac{D}{I}}T\,,$$

$$2\pi\sqrt{\frac{I}{D}} = T\,.$$

Either way, we reached the formula $T = 2\pi\sqrt{I/D}$; expressed using given variables, it is

$$T = 2\pi\sqrt{\frac{I}{D}} = 2\pi\sqrt{\frac{ma^2/3}{Kv^2a^2b/2}} = 2\pi\sqrt{\frac{2m}{3Kv^2b}}\,.$$

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Problem FE . . . washing a chopping board

Imagine you are washing a chopping board in the sink. You turn it at an angle $\alpha = 45^\circ$ relative to the ground (the shorter edge touches the bottom of the sink) and let the water fall on it. Assume that the water bounces in all directions in the plane of the chopping board at $v = 45\,\mathrm{cm{\cdot}s^{-1}}$. What percentage of the top of this kitchen utensil is wetted by water if its dimensions are $h = 27\,\mathrm{cm}$ and $d = 17\,\mathrm{cm}$ and the water drips on its center?

Jarda has a tendency to formulate a problem while engaged in any activity.

At each point on the chopping board, a gravitational acceleration component of magnitude $g\sin\alpha$ acts on the water down the board. So the situation is analogous to a safety parabola – we are investigating all points where water may be after bouncing off the center of the chopping board.

We introduce coordinates on the board by placing the center in the middle of the bottom edge, with the x axis running horizontally and the y axis running perpendicular to it along the surface of the chopping board.

Then, the area which the water reaches lies below the curve

$$y = \frac{h}{2} + \frac{v^2}{2g\sin\alpha} - \frac{x^2 g\sin\alpha}{2v^2}\,.$$

We must still investigate if the safety parabola ever crossed the bottom edge. We find these points by putting $y = 0$ in the previous equation as

$$x = \pm\sqrt{\frac{hv^2}{g\sin\alpha} + \frac{v^4}{g^2\sin^2\alpha}} = 9.3\,\mathrm{cm} > \frac{d}{2} = 8.5\,\mathrm{cm}\,.$$

We found that the parabola does not cross the bottom edge of the chopping board anywhere. That makes it easy to calculate the area using an integral with limits from $-d/2$ to $d/2$

$$\int_{-d/2}^{d/2}\left(\frac{h}{2} + \frac{v^2}{2g\sin\alpha} - \frac{x^2 g\sin\alpha}{2v^2}\right)\mathrm{d}x = \frac{h}{2}d + \frac{v^2}{2g\sin\alpha}d - \frac{d^3 g\sin\alpha}{24v^2}\,.$$

Since we are asking for a fraction, we need to determine the ratio of this area to the area of the whole board, so we express the solution to the problem as

$$p = \frac{1}{2} + \frac{v^2}{2gh\sin\alpha} - \frac{d^2 g\sin\alpha}{24hv^2} = 0.40 = 40\,\%\,.$$

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Problem FF ... Arda

The world from the book The Silmarillion is called Arda. Instead of a sphere, this world is shaped like a disk with a radius much larger than its thickness. However, the gravitational acceleration on the surface, in the center of the disk, is the same as on Earth. Find the area density of Arda.

Jarda's memory of a succesful camp with a box full of experimental equipment.

We use the analogy of Gauss's law from electrostatics but with different constants. In electrostatics, in the surroundings of a large plate with a surface charge density σ, the intensity is $E = \sigma/(2\varepsilon)$. The intensity E corresponds to the gravitational acceleration g and ε is analogous to the gravitational constant.

From a comparison of Newton's law of universal gravitation and Coulomb's law, we find the analogy $\varepsilon = 1/(4\pi G)$. The area density of Arda is then given as

$$\sigma = \frac{g}{2\pi G} = 2.3\cdot 10^{10}\,\mathrm{kg\cdot m^{-2}}\,.$$

If we were to consider the density of the material of the world to be $\rho = 5\,000\,\mathrm{kg\cdot m^{-3}}$, its thickness would have to be $d = \sigma/\rho = 4\,700\,\mathrm{km}$. However, for the approximation from the problem statement to be valid, i.e. that the radius of the disk is much larger than its thickness, this radius would have to be many times larger than, for example, the radius of our Earth.

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Problem FG ... boring vacation

This year, Fykosaurus went on holiday at Christmas. Since he became bored of lying by the sea all day at an exotic destination, he explored the local technical sights. In one museum he found a mathematical pendulum that was swinging in a south to north direction at the time of his arrival. When he left the building, eight and a half hours later, it was swinging in a west to east direction. On what latitude did Fykosaurus spend his vacation?

Jarda continues with Foucault pendulum problems.

You certainly have heard of the Foucalt pendulum, which was used to demonstrate the rotation of the Earth on its axis in the eighteenth century. To find the solution of the problem, we need to determine the angular velocity at which the plane of the pendulum rotates. In advance, we can disclose that it is $\omega_1 = \Omega \sin\lambda$, where $\Omega = 2\pi(24\,\mathrm{h})$ is the angular velocity of Earth's rotation and λ is the latitude at which is Fykosaurus located.

Let's introduce the Cartesian coordinate system in the museum. Let the z axis point perpendicular to the surface, the x axis point east, and the y axis point north. The rotation of the plane of the pendulum oscillations is caused by the Coriolis force, which appears in rotating reference frames. Our established frame is certainly one of such systems, which is why this force appears here. It acts on objects that are moving radially in the direction of rotation, namely tangentially. In addition to this force, the force of gravity and the force of the hinge also act on the pendulum.

The angular velocity vector of the Earth's rotation in a given reference frame has components

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ \Omega\cos\lambda \\ \Omega\sin\lambda \end{pmatrix} .$$

We expres the Coriolis force as

$$\mathbf{F}_\mathrm{C} = 2m\mathbf{v} \times \boldsymbol{\omega} ,$$

where m is the mass of the pendulum and $\mathbf{v}$ is its velocity vector in our reference frame. If we neglect motion in the z axis (mathematical pendulums have minimum vertical deviations), we can write the force vector as

$$\mathbf{F}_\mathrm{C} = 2m \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \Omega\cos\lambda \\ \Omega\sin\lambda \end{pmatrix} = 2m\Omega \begin{pmatrix} \dot{y}\sin\lambda \\ -\dot{x}\sin\lambda \\ \dot{x}\cos\lambda \end{pmatrix} ,$$

where the dots indicate the velocity in the x and y axes. From now on, let us restrict ourselves to motion in the x, y plane. There is also a gravitational force component. The mathematical pendulum behaves analogously to a linear harmonic oscillator and its motion in the x, y plane is determined by the force

$$\mathbf{F}_{x,y} = -m\omega_g^2 \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} ,$$

where $\omega_g = \sqrt{g/L}$ is the angular frequency of oscillations and x and y are the deflections of the pendulum from the equilibrium position in both perpendicular directions.

The resultant of the two forces gives the equations of the motion of the pendulum. We write them for the components x and y as

$$m\ddot{x} = -m\omega_g^2 x + 2m\Omega\sin(\lambda)\,\dot{y}\,,$$
$$m\ddot{y} = -m\omega_g^2 y - 2m\Omega\sin(\lambda)\,\dot{x}\,.$$

To solve this system, we try the following trick: we first truncate both equations, then multiply the second one by the complex unit i and finally add them. We get

$$\ddot{x} + i\ddot{y} = -\omega_g^2\left(x + iy\right) + 2\Omega\sin(\lambda)\left(\dot{y} - i\dot{x}\right)\,.$$

From the last bracket we extract $-i$ and thanks to the relation $i^2 = -1$ we can express the whole equation as

$$\ddot{u} = -\omega_g^2 u - 2i\Omega\sin(\lambda)\,\dot{u}\,,$$

where $u = x + iy$ is our new complex variable that represents the position of the pendulum in the Gaussian plane. We have done nothing more than write the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as a single complex number, giving us only one equation out of two.

This equation is analogous to that of a damped harmonic oscillator, where the resistant force is proportional to the speed of motion. The solution are damped oscillations, where the cosine waveform is exponentially damped. So we try to write the solution of the equation as

$$u = u_0\exp(i\omega_1 t)\cos(\omega_2 t)\,,$$

where u_0 is some (again complex) amplitude. By substituting into the differential equation, we get

$$\begin{aligned}&-u_0\omega_1^2\exp(i\omega_1 t)\cos(\omega_2 t) - i\omega_1\omega_2 u_0\exp(i\omega_1 t)\sin(\omega_2 t) - i\omega_1\omega_2 u_0\exp(i\omega_1 t)\sin(\omega_2 t) -\\ &-u_0\omega_2^2\exp(i\omega_1 t)\cos(\omega_2 t) = -\omega_g^2 u_0\exp(i\omega_1 t)\cos(\omega_2 t) - 2i\Omega\sin(\lambda)\,u_0 i\omega_1\exp(i\omega_1 t)\cos(\omega_2 t)\\ &+2i\Omega\omega_2\sin(\lambda)\,u_0\exp(i\omega_1 t)\sin(\omega_2 t)\,.\end{aligned}$$

After truncation by the factor $u_0\exp(i\omega_1 t)$ and some adjustments, we get

$$\begin{aligned}&-\omega_1^2\cos(\omega_2 t) - 2i\omega_1\omega_2\sin(\omega_2 t) - \omega_2^2\cos(\omega_2 t) =\\ &= -\omega_g^2\cos(\omega_2 t) + 2\Omega\omega_1\sin(\lambda)\cos(\omega_2 t) + 2i\Omega\omega_2\sin(\lambda)\sin(\omega_2 t)\,.\end{aligned}$$

This equation must be satisfied at all times t, thus the sines and cosines must be equal separately, and so we can split it into two equations

$$\begin{aligned}-2i\omega_1\omega_2\sin(\omega_2 t) &= 2i\Omega\omega_2\sin(\lambda)\sin(\omega_2 t) &&\Rightarrow\quad \omega_1 = -\Omega\sin(\lambda)\,,\\ -\omega_1^2\cos(\omega_2 t) - \omega_2^2\cos(\omega_2 t) &= -\omega_g^2\cos(\omega_2 t) + 2\Omega\omega_1\sin(\lambda)\cos(\omega_2 t) &&\Rightarrow\quad \omega_2 = \sqrt{\omega_g^2 + \omega_1^2}\,.\end{aligned}$$

Foucalt pendulums have a suspension length in the tens of meters, which corresponds to ω_g in the order of tithes of Hz. This is about 4 orders of magnitude more than Ω, so we can neglect the ω_1^2 term in the square root relative to ω_g and substitute u into our expression.

$$u = u_0\exp(-i\Omega\sin(\lambda)\,t)\cos(\omega_g t)\,.$$

Therefore, the pendulum swings with angular frequency ω_g, but also makes a rotational motion with angular frequency $\Omega\sin(\lambda)$ (exponential with complex exponent). The minus sign in the exponential even tells us in which direction the plane of oscillation will rotate.

According to the problem statement, in the time $T = 8.5\,\mathrm{h}$, the plane of oscillation has rotated by $90\,^\circ$, which corresponds to

$$T\omega_1 = \frac{T2\pi\sin\lambda}{24\,\mathrm{h}} = 90\,^\circ = \frac{\pi}{2} \quad \Rightarrow \quad \lambda = \arcsin\left(\frac{1}{4}\frac{24\,\mathrm{h}}{8.5\,\mathrm{h}}\right) \doteq 45\,^\circ\,.$$

Let us also note that no other solution (rotation by $270\,^\circ$ or more) is possible.

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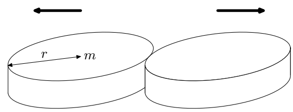
Problem FH ... separable couple



Two identical cylindrical magnets of mass $m = 7.5\,\mathrm{g}$ and radius $R = 1.1\,\mathrm{cm}$ are positioned horizontally in a manner that they repel each other while touching. Upon release, they move to a distance $x = 11\,\mathrm{cm}$ between their centers. What is the magnetic moment μ of each magnet? The coefficient of friction between the magnets and the substrate is $f = 0.35$. Consider the dipole-dipole interaction between the magnets. *Thanks to Jirka, Jarda became an expert on magnets.*

What is the magnetic force between the magnets? We can use the knowledge from magnetostatics, according to which the structure of the magnetic field of a dipole is indistinguishable from the electrostatic field of an electric dipole. Thus, we can convert the force calculation into an electrostatics calculation. Let's consider substituting magnetic dipoles with electric dipoles. For them, $p = q\delta x$, where q is the charge of the individual charges in the dipole, and δx is their mutual distance. For example, since magnets repel each other, they must have both positive charges at the top and both negative charges at the bottom. The electrostatic force by which one magnet repels the other is

$$F = \frac{2}{4\pi\varepsilon_0}\left(\frac{q^2}{r^2} - \frac{rq^2}{\left(r^2 + (\delta x)^2\right)^{\frac{3}{2}}}\right)\,,$$

where r is the distance between the centers of the magnets. Since we are considering dipoles, $\delta x \ll r$ and we can develop a Taylor series of the second order

$$F = \frac{2}{4\pi\varepsilon_0}\frac{q^2}{r^2}\left(1 - \frac{1}{\left(1 + \left(\frac{\delta x}{r}\right)^2\right)^{\frac{3}{2}}}\right) \approx \frac{2}{4\pi\varepsilon_0}\frac{q^2}{r^2}\left(1 - 1 + \frac{3}{2}\left(\frac{\delta x}{r}\right)^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}p^2\,.$$

Now, we must move from the electrostatic force to the magnetic force. We replace the electric dipole with a magnetic μ and replace the proportionality constant $1/4\pi\varepsilon$ with $\mu_0/4\pi$. The magnitude of the force is then

$$F = \frac{3\mu_0}{4\pi}\frac{\mu^2}{r^4}\,,$$

where r is the distance between the centers of the magnets. The potential energy of the system at the origin is therefore

$$E_\mathrm{i} = -\int_\infty^{2R} F\,\mathrm{d}r = \frac{\mu_0}{4\pi}\frac{\mu^2}{(2R)^3}\,.$$

The portion of this energy transforms into work executed by frictional forces. If we displace one of the magnets from its initial position by $x/2 - R$, then this work is equal to

$$W = 2mgf\left(\frac{x}{2} - R\right) = mgf(x - 2R)\,.$$

After the displacement, the mutual distance of the centers of the magnets is x, and their potential energy is

$$E_\mathrm{f} = -\int_\infty^{x} F\,\mathrm{d}r = \frac{\mu_0}{4\pi}\frac{\mu^2}{x^3}\,.$$

Thus, from the law of conservation of energy, we get the relation

$$E_\mathrm{f} + W = E_\mathrm{i} \quad \Rightarrow \quad \mu = \sqrt{\frac{4\pi mgf(x-2R)}{\mu_0}\frac{x^3\,(2R)^3}{x^3 - (2R)^3}} = 0.49\,\mathrm{A{\cdot}m^2}\,.$$

According to the assignment we considered dipole-dipole interaction between magnets. In the case of real magnets, with this approximation we can get because magnets are usually made of magnetized material that has non-zero dimensions. At least in the case of homogeneous magnetization, we can describe magnets in terms of a plane magnetic charge and then calculate the force as in electrostatics for plane electric charges. For a bar magnet whose height is much greater than its diameter, we then get that at large distances from the magnet the magnetic field corresponds to that of a dipole, since we approximate the surface charge on the bases by point charges.

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Problem GA ... Once again, there's one left!

Jarda was bowling with his friends, but as usual, he was not doing well. Finally, it looked like he was going to get a strike, but once again, there was one pin left standing. By this time Jarda was getting really upset, so he put so much energy into his second throw that the ball was moving at $0.7c$. Surprisingly, he hit the bowling pin, there was an inelastic collision, and both objects merged into one. What is their combined mass after stopping if the ball had a mass of $M = 7.0\,\mathrm{kg}$ before the throw and the bowling pin had mass $m = 1.5\,\mathrm{kg}$?

Jarda is scared of getting fat at the bowling alley.

Since the ball is moving at a very high velocity, we have to consider the special theory of relativity. We still have the law of conservation of momentum, and although it is an inelastic collision, the law of conservation of energy. We are going to assume that the energy of the collision is converted into mass.

However, we must allow for relativistic momenta and other adjustments. If the body has an invariant mass (rest mass) of M and a velocity of v, its total energy (i.e. including the invariant mass) is

$$E = \frac{Mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\,.$$

The relativistic momentum is

$$p = \frac{Mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\,.$$

This is actually how we defined the energy and momentum before the collision, when M is the mass of the sphere and $v = 0.7c$ is its velocity.

Let's denote by μ the invariant mass of the combination of the bowling pin and the ball (whatever that looks like) after the collision and by u their combined velocity. Then the momentum after the collision is

$$p = \frac{\mu u}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

and the energy is

$$E = \frac{\mu c^2}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}\,.$$

The momentum before the collision must be equal to the momentum after the collision. To the kinetic energy of the sphere, we also add the rest energy of the bowling pin. We get a pair of equations

$$\frac{Mv}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{\mu u}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

and

$$\frac{Mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} + mc^2 = \frac{\mu c^2}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}\,.$$

The right-hand side of the second equation is substituted into the first equation, and then we can express u as

$$\frac{v}{1 + \frac{m}{M}\sqrt{1 - \left(\frac{v}{c}\right)^2}} = u\,.$$

Substitute back into the previous equation and we have

$$\mu = M\sqrt{\frac{\left(1 + \frac{m}{M}\sqrt{1 - \frac{v^2}{c^2}}\right)^2 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}} = 8.98\,\mathrm{kg}\,.$$

It is clear that the rest mass after the collision is greater than the sum of the rest masses before the collision.

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Problem GB ... 21 zentimeter

Neutral hydrogen in interstellar space can emit a photon with wavelength $\lambda = 21\,\mathrm{cm}$, when it transitions from a higher energetic state to a lower one. The half-life in a higher energetic state is $\tau = 1.1 \cdot 10^{7}$ years. In the sky, we have identified a spherical source of this radiation, from which flux density $F = 1.5 \cdot 10^{-24}\,\mathrm{W{\cdot}m^{-2}}$ is reaching us. The diameter of the source is $d = 15\,\mathrm{ly}$ and its distance from Earth is $R = 21\,000\,\mathrm{ly}$. Determine the mass density of hydrogen in the cloud (source of radiation). *Jarda was listening to a German song.*

The flux density F corresponds to

$$n = \frac{F\lambda}{hc}$$

number of photons incident on one square meter per second. If we denote the distance of the source from the Earth by R, then it emits

$$A = 4\pi R^2 n$$

photons per second, which is the activity of the whole source. From this, we calculate the number of hydrogen nuclei in the cloud

$$N = A\tau = 4\pi R^2 n\tau\,.$$

Here, we need to think more deeply. The number N gives the number of atoms that can emit a photon, so they must be in a higher energetic state. The transition, with a wavelength of 21 cm, transpires when the mutual orientations of spins of the electron and the proton in a single atom undergo alteration. Applying Boltzmann's distribution, it is essential to ascertain the probability that an atom resides in a higher energetic state. Given the temperature on the order of kelvins and the exceedingly small energy associated with the transition, all microstates are essentially filled with equal probability ($k_B T \gg hc/\lambda$). That is because the hydrogen gas hosting the transition would have at least the temperature of the cosmic microwave background, $T = 2.7\,\mathrm{K}$ (though more likely a higher temperature, as hydrogen clouds typically exist within galaxies and are heated by radiation from stars). The temperature of our hydrogen transition is $T_\lambda = hc/(\lambda k_B) = 0.068\,5\,\mathrm{K}$.

We're nearing completion. The total number of atoms should be approximately twice as large as N. However, it is essential to note that Boltzmann's distribution provides the probability of a single microstate, not an entire energy level. According to the principles of quantum mechanics, the spins of an electron and a proton can combine in four different ways. Out of these, three have higher energy, and only one has lower energy. Consequently, according to Boltzmann's distribution, the state of an atom with higher energy occurs three times as often as a state with lower energy. Hence, the total number of hydrogen atoms is $4/3N$.

We multiply all the hydrogen nuclei in the cloud by the mass of each of them, m_u. To find the density, we divide the total mass by the volume of the spherical cloud, and we get

$$\rho = \frac{m_\mathrm{u}}{\frac{4}{3}\pi\left(\frac{d}{2}\right)^3}\frac{4N}{3} = \frac{32 m_\mathrm{u} R^2 F\lambda\tau}{hcd^3} = 4.1 \cdot 10^{-22}\,\mathrm{kg{\cdot}m^{-3}}\,.$$

On our timescale, we can regard the fraction of atoms in the higher energetic states as constant. That is because new excited atoms are generated continually in a hydrogen cloud. It happens due to random collisions between them, which may be rare, but on the other hand, the excited states of hydrogen have quite a long half-life. The thermal kinetic energy of hydrogen atoms is orders of magnitude higher than the energy required for spin flipping, so there is enough energy for continuous excitation.

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Problem GC ... Pew Pew

Tomáš and Viktor have elevated their airsoft matches to a new dimension: space. In this scenario, what is the minimum distance from Viktor's laser gun at which Tomáš must commence his uniformly accelerated motion to ensure that a laser pulse fired directly at him cannot reach him? We assume that Tomáš and his vehicle have a rest mass of m_0 and are subjected to acceleration by a constant force f. Tomáš initiates his movement simultaneously with the shot from the laser gun, as observed from Viktor's reference frame at rest.

Marek J. found out that it is possible to outrun light.

Despite the conventional understanding that the speed of light in a vacuum is the ultimate attainable speed, an intriguing anomaly occurs in this scenario – Tomáš can evade light indefinitely. We will defer an intuitive explanation of this phenomenon until later. For now, let us focus on calculating this counter-intuitive fact.

We can view this problem as one-dimensional. In the case of relativistic mechanics, the equation

$$f = \frac{\mathrm{d}p}{\mathrm{d}t}$$

still holds, but with one important change related to momentum: $p = mv$, where m is the so-called "relativistic" mass/energy, $m = \gamma m_0$ with the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. We see that m changes with changing velocity v. Therefore, we need to solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{cm_0 v}{\sqrt{c^2 - v^2}}\right) = f\,,$$

where only the velocity of Tomáš v depends on time. Straightforward integration and subsequently solving the quadratic equation for v gives

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = \frac{ft}{m_0\sqrt{1 + \left(\frac{ft}{m_0 c}\right)^2}}\,. \tag{9}$$

Once again, we proceed with integration, this time addressing the integral of the right-hand side in equation (9). We employ the substitution $ft/m_0 c = \sinh u$ to solve the integral. Consequently, we derive the trajectory for Tomáš as

$$x = \frac{m_0 c^2}{f}\sqrt{1 + \left(\frac{ft}{m_0 c}\right)^2} + C\,,$$

where we find the value of the integration constant C from the initial condition, which is that at the time $t = 0$, Tomáš is at the distance d from Viktor. Then $C = d - m_0 c^2/f$.

If the ray of light could reach Tomáš, an intersection of their trajectories or a time of intersection would be calculable

$$ct = \frac{m_0 c^2}{f}\left[\sqrt{1+\left(\frac{ft}{m_0 c}\right)^2} - 1\right] + d\,. \tag{10}$$

We should check how this condition depends on d. The easiest way to start is probably trying $d = m_0 c^2/f$. Then, from the equation (10)

$$\frac{ft}{m_0 c} = \sqrt{1+\left(\frac{ft}{m_0 c}\right)^2}\,,$$

which cannot happen for any time t (the right-hand side is always larger). Thus $d = m_0 c^2/f$ is a distance from which Viktor cannot ever hit Tomáš. However, we need to find the minimum distance, so we consider $d = m_0 c^2/f - \varepsilon$, where $\varepsilon > 0$ is typically very small. Also, we should realize that we only need to consider $\varepsilon < m_0 c^2/f$, since we would get $d < 0$ otherwise. After substituting for d in the equation (10), we get an expression for time

$$t = \frac{1 - \frac{f^2}{m_0^2 c^4}\varepsilon^2}{\frac{2f^2}{m_0^2 c^3}\varepsilon}\,,$$

and considering just time with a positive sign, we get a solution if $0 < \varepsilon < m_0 c^2/f$ (the condition of positive numerator). In simpler terms, for any reduction in distance from $d = m_0 c^2/f$, there exists a corresponding time at which the laser ray reaches Tomáš. Thus, we have demonstrated that $d = m_0 c^2/f$ is the minimum distance we sought to determine. This conclusion aligns with an alternative approach of considering the geometric properties of a hyperbola.

Finally, as promised, here is an intuitive explanation akin to Zeno's paradox. Picture light, much like Achilles striving to catch a turtle, reaching a point from which Tomáš has already moved. However, the situation differs from Achilles and the turtle. Achilles and the turtle move with uniform velocities, and the distance by which Achilles misses the turtle diminishes sufficiently at every "step" for Achilles to reach the turtle in a finite time. In the case of light and Tomáš, this distance does not decrease rapidly enough because Tomáš, unlike light (and the turtle), accelerates at every step. Consequently, light does not reach Tomáš in a finite time but only at infinity.

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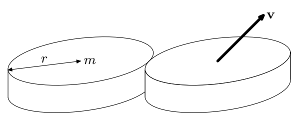
Problem GD ... inseparable couple

Two identical cylindrical magnets of mass $m = 7.5\,\mathrm{g}$, radius $r = 1.1\,\mathrm{cm}$, and dipole moment $\mu = 1.1\,\mathrm{A{\cdot}m^2}$ are placed on a perfectly smooth horizontal surface. They touch each other in a way that induces attraction, causing them to remain stuck together. The question at hand is to determine the maximum velocity v relative to the base that can be imparted to one of the magnets perpendicular to the line connecting their centers, ensuring that the magnets remain firmly attached and do not separate.



You can neglect friction but not the dipole-dipole interaction between the magnets.

Jarda did not want to leave his girlfriend.

First, let us shift to the reference frame of the magnets' center of mass, where both magnets are orbiting the point where they are touching at all times. There, the velocity of the center of each magnet with respect to the center of mass is $v/2$.

In the problem labeled FH – "separable couple" – we showed that the attractive force between the magnets is equal to

$$F_\mathrm{m} = \frac{\mu_0}{4\pi} \frac{3\mu^2}{(2r)^4} \,,$$

where $2r$ is the distance between the centers of the magnets. To be precise, we have derived that result for repulsive force, but it is straightforward to figure out that when the magnets attract, the force has the same magnitude and opposite direction. In any case, this force has to be greater than the centrifugal force, which value is

$$F_\mathrm{o} = m\omega^2 r = m\left(\frac{v}{2r}\right)^2 r \,.$$

From the condition where both forces are equal in magnitude, we obtain

$$\frac{\mu_0}{4\pi} \frac{3m^2}{(2r)^4} = m\left(\frac{v}{2r}\right)^2 r \quad \Rightarrow \quad v = \sqrt{\frac{3\mu_0}{16\pi} \frac{\mu^2}{mr^3}} = 3.0\,\mathrm{m{\cdot}s^{-1}} \,.$$

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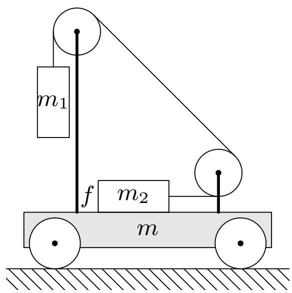
Problem GE ... self-propulsion

The figure depicts a trolley with two weights connected by a rope and a system of pulleys. The masses of the cubical weights are $m_1 = 1.5\,\mathrm{kg}$, $m_2 = 1.0\,\mathrm{kg}$ and the mass of the trolley is $m = 3.0\,\mathrm{kg}$. The coefficient of friction between the cuboid with mass m_2 and the surface of the trolley is $f = 0.40$. The pulleys and the rope are massless and frictionless. What will the acceleration of the trolley be once both weights and the trolley are released to move freely? We assume that the system rapidly reaches its steady state. The accompanying image depicts the trolley and the two weights prior to their release.



Jarda tends to let Jindra write the solutions to his problems.

First, let's try to guess the direction of the trolley's movement. The cuboid with mass m_2 starts moving to the right since the rope pulls it with force T. This force is transferred to a pulley, where it (due to the law of action and reaction) accelerates the trolley to the left. We will see later that multiple forces are acting on the trolley, but we assume that the force T dominates.

Let's denote the acceleration of the trolley by A. When the sign of A is positive, the trolley accelerates to the left. The acceleration of the trolley causes inertial forces to act on the cuboids in the trolley's frame of reference. Thus, let us switch to the trolley's frame of reference and write down the equations of motion of the cuboids. The rope is transmitting a tension force denoted as T. As the two cuboids are linked by the rope, they experience the same acceleration denoted as a. The trolley's acceleration to the left suggests that the inertial force is acting to the right. Newton's second law for the cuboid with mass m_2 in the horizontal direction states that:

$$m_2 a = T + m_2 A - f m_2 g\,, \tag{11}$$

where $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$ is the acceleration due to gravity.

Since the trolley is accelerating, the cuboid with mass m_1 does not hang directly downward; instead, it tilts at an angle α to the right of the vertical. The angle α satisfies

$$\tan\alpha = \frac{A}{g}\,. \tag{12}$$

The forces acting on the cuboid are the tension force T exerted by the rope, gravity $m_1 g$ pulling it downwards, and the inertial force $m_1 A$ acting to the right. The cuboid m_1, therefore, possesses acceleration a in the direction of the rope, just like the cuboid m_2. Newton's second law in the vertical direction says

$$m_1 a\cos\alpha = m_1 g - T\cos\alpha \tag{13}$$

and in the horizontal direction, it is

$$m_1 a\sin\alpha = m_1 A - T\sin\alpha\,. \tag{14}$$

First, let us check that the cuboids start moving despite friction when the trolley is released. In that case, $A = 0$ and $\alpha = 0$. We need to solve the system of two equations

$$m_2 a = T - f m_2 g\,,$$
$$m_1 a = m_1 g - T\,,$$

with two unknowns T and a. From the first equation, we express

$$T = m_2 a + f m_2 g\,,$$

substitute into the second equation and express the acceleration a

$$m_1 a = m_1 g - m_2 a - f m_2 g\,,$$
$$a = \frac{m_1 - f m_2}{m_1 + m_2} g\,.$$

The cuboids start moving only when the resulting acceleration a is positive. In our case, $m_1 = 1.5\,\mathrm{kg} > f m_2 = 0.4\,\mathrm{kg}$, so the system really starts moving.

We can return to the system of equations (11), (13), (14) describing the motion of the cuboids in the reference frame of the accelerating trolley. They contain four unknowns T, α, a, A. The equation (12) is a linear combination of equations (13) and (14), providing no additional information. Hence, a fourth equation is necessary, namely Newton's second law for the trolley. The reaction force from the rope T is pulling the trolley to the left. At the left pulley, a force $T\sin\alpha$ is pulling the trolley to the right. The friction force $f m_2 g$ from the cuboid m_2 pulls the trolley to the right. Tension forces from the rope segment between the pulleys cancel each other out, thus not influencing the trolley's acceleration. Additionally, there is the inertial force mA acting to the right in the reference frame of the trolley. Since the acceleration of the trolley is zero, we derive the equation

$$0 = T - T\sin\alpha - fm_2g - mA.$$

Now, we have a system of four equations

$$\begin{aligned}
m_1 a\cos\alpha &= m_1 g - T\cos\alpha\,,\\
m_1 a\sin\alpha &= m_1 A - T\sin\alpha\,,\\
m_2 a &= T + m_2 A - fm_2 g\,,\\
0 &= T - T\sin\alpha - fm_2 g - mA\,,
\end{aligned}$$

with four unknowns T, α, a, A. From the equation (12), we can express the acceleration $A = g\tan\alpha$ to get rid of one unknown

$$\begin{aligned}
m_1 a\cos\alpha &= m_1 g - T\cos\alpha\,,\\
m_2 a &= T + m_2 g\tan\alpha - fm_2 g\,,\\
0 &= T(1-\sin\alpha) - fm_2 g - mg\tan\alpha\,.
\end{aligned}$$

If we find the angle α, we can express the acceleration A from (12). Therefore, we express

$$a = \frac{g}{\cos\alpha} - \frac{T}{m_1}$$

from the first equation and plug it into the next two equations to get rid of another unknown

$$\begin{aligned}
\frac{m_2 g}{\cos\alpha} - \frac{m_2}{m_1}T &= T + m_2 g\tan\alpha - fm_2 g\,,\\
0 &= T(1-\sin\alpha) - fm_2 g - mg\tan\alpha\,.
\end{aligned}$$

Now, we express the tension force

$$T = \frac{m_2 g - m_2 g\sin\alpha + fm_2 g\cos\alpha}{\left(1 + \frac{m_2}{m_1}\right)\cos\alpha}\,,$$

from the first equation and substitute it into the second one, getting one equation with one unknown α, which we need to solve numerically

$$0 = m_2 g(1-\sin\alpha)\frac{1 - \sin\alpha + f\cos\alpha}{\left(1 + \frac{m_2}{m_1}\right)\cos\alpha} - fm_2 g - mg\tan\alpha\,.$$

We are going to solve it using an iterative method, where we put the unknown on the left-hand side and get some function of the unknown at the right-hand side

$$\alpha = f(\alpha)\ .$$

We use the initial estimate of the solution $\alpha = \alpha_0$, which we plug into the function and calculate the second estimate of the solution $\alpha_1 = f(\alpha_0)$. Every subsequent estimate of the solution α_{i+1} is calculated from the previous estimate as

$$\alpha_{i+1} = f(\alpha_i)\ . \tag{15}$$

If we are lucky, the values α_i may converge toward a singular value. At that point, it is within our discretion to determine when we are content with the precision of the solution α_i and decide to conclude the iteration process. If the sequence (15) does not converge, we need to change up the equation, express the unknown using another function $\alpha = f'(\alpha)$ and iterate again.

We are going to use the function

$$\alpha = \arctan\left(\frac{m_2}{m}(1-\sin\alpha)\frac{1-\sin\alpha + f\cos\alpha}{\left(1+\frac{m_2}{m_1}\right)\cos\alpha} - f\frac{m_2}{m}\right).$$

With the initial estimate $\alpha_0 = 4.0\,^\circ$, in a few steps, we converge to the solution $\alpha = 5.78\,^\circ$. With help of the equation (12), we then calculate the acceleration of the trolley $A = g\tan\alpha = 0.993\,\mathrm{m{\cdot}s^{-2}} \doteq 0.99\,\mathrm{m{\cdot}s^{-2}}$.

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Problem GF ... decaying problem

The radioactive nuclei of fykosium are obtained from a nuclear reaction that produces P nuclei per second. If at time $t = 0$ we have N_F nuclei of fykosium, what will be their activity at time T, i.e., how many of them will decay per second? We know that the half-life of fykosium is $T_{1/2}$. *Marek was going through a spontaneous personality decay.*

The number of decaying nuclei is proportional to the total number of nuclei, with a proportionality constant $\lambda = \ln 2/T_{1/2}$ (with the unit $\mathrm{s^{-1}}$). At the same time, the nuclei are being created at the rate P. Hence, the following holds for the derivative of the number of nuclei N

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N + P\,,$$

where λ is the decay constant of fykosium.

Then

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} + \lambda N &= P\,,\\
\mathrm{e}^{\lambda t}\frac{\mathrm{d}N}{\mathrm{d}t} + \lambda\mathrm{e}^{\lambda t}N &= \mathrm{e}^{\lambda t}P\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda t}N\right) &= \mathrm{e}^{\lambda t}P\,,\\
\mathrm{e}^{\lambda t}N &= \int \mathrm{e}^{\lambda t}P\,\mathrm{d}t\,,\\
N(t) &= C\mathrm{e}^{-\lambda t} + \frac{P}{\lambda}\,,
\end{aligned}$$

where C is an unknown integration constant, which we determine from the fact that $N(0) = N_\mathrm{F}$. Consequently

$$C = N_\mathrm{F} - \frac{P}{\lambda}$$

and

$$N(t) = N_\mathrm{F}\mathrm{e}^{-\lambda t} + \frac{P}{\lambda}\left(1 - \mathrm{e}^{-\lambda t}\right)\,.$$

The activity of a sample is the number of decays with respect to time. Let's therefore calculate how the number of nuclei changes over time

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N_\mathrm{F} \mathrm{e}^{-\lambda t} + P \mathrm{e}^{-\lambda t} + (P - P) = -\lambda N_\mathrm{F} \mathrm{e}^{-\lambda t} - P\left(1 - \mathrm{e}^{-\lambda t}\right) + P\,.$$

When we compare this result with the first equation, we see that the first two terms account for the decay, while the last one determines the production of nuclei from the reaction and does not contribute to the activity. Finally, we note that $T_{1/2} = \ln 2/\lambda$ and after adjusting the exponentials we obtain

$$A(T) = \frac{\ln 2}{T_{1/2}} N_\mathrm{F} 2^{-\frac{T}{T_{1/2}}} + P\left(1 - 2^{-\frac{T}{T_{1/2}}}\right)\,.$$

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