Solutions of the Fyziklani 2021

15^{th} year

Problem M.1 ... a pram 5 points

A pram (baby carriage) weighs $M = 7.0\,\mathrm{kg}$. Its weight is uniformly distributed over four wheels. Find the ratio of the force which a mother must exert on a pram with a baby weighing $m_1 = 7.0\,\mathrm{kg}$ in order to move it, to this force for a pram with a baby weighing $m_2 = 3.5\,\mathrm{kg}$. The main source of resistance is located at the points of contact between the wheels and the ground.

Danka is wondering about maternity.

The rolling resistance is given as

$$F_\mathrm{r} = fF_\mathrm{n}\,,$$

where f is coefficient of friction[1] and F_n is the normal force, that pushes the wheel against the ground, which, in our case, equals $F_\mathrm{n} = F_g = (M + m_{1,2})\,g$. The fact, that the weight is distributed over all four wheels, does not affect the friction force in any way. For ratio of friction forces we get

$$\frac{F_{\mathrm{r},1}}{F_{\mathrm{r},2}} = \frac{fg\,(M+m_1)}{fg\,(M+m_2)} \doteq 1.33\,.$$

The mother must exert a force 1.33 times greater while pushing a heavier child.

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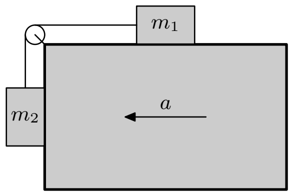
Problem M.2 ... pulleys on a railway car 5 points



Consider a railway car in the shape of a cuboid which moves with a constant acceleration a. On its upper surface, there is a cuboid (mass $m_1 = 15\,\mathrm{kg}$) attached by a rope to another cuboid (mass $m_2 = 10\,\mathrm{kg}$), which is hanging freely at the front side of the car. Find the value of the acceleration a which prevents those cuboids from accelerating (with respect to the car), assuming that there is zero friction in the system. *Lego likes pulleys.*

To keep the hanging cuboid from accelerating, the vertical component of the net force acting on it must be zero. The horizontal component, which results from the acceleration of the system, is compensated by normal force exerted by the front surface of the car. The gravitational force acting on the cuboid is simply $F_2 = m_2g$. The only force that could compensate it is the force exerted by the rope. Therefore, this force must pull the cuboid upwards and its magnitude must be $T = F_2$.

The second cuboid is pulled by the same force towards the front of the car. This is the only real force exerted on the cuboid in the horizontal direction. To keep it from accelerating (with respect to the car), this force must cause it to accelerate forward with the acceleration a – therefore, its magnitude is $F_1 = m_1a$. The gravitational force cancels out with normal force exerted by the roof and the only force left is the force pulling the rope, so $F_1 = T$.

[1] This is often given as the ratio of the arm of rolling friction (also called rolling friction coefficient) and the radius of the wheel, but that is not entirely correct.

Putting it all together, we get the desired acceleration of the car

$$a = \frac{m_2}{m_1} g \doteq 6.5\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem M.3 ... a restless washing machine 5 points

In a washing machine, there is a cylindrical drum with laundry, whose axis is horizontal. We approximate the wet laundry by a rigid body with a mass $m = 3.5\,\mathrm{kg}$ and with the centre of mass located at a distance $r = 4.9\,\mathrm{cm}$ from the axis of rotation of the drum. The drum rotates at 421 rpm. What is the minimum weight of the washing machine needed to ensure that it does not bounce? Assume that the washing machine can only move in the vertical direction.

Jarda was distracted from creating problems by a washing machine.

The laundry is maintained on the circular trajectory by the centripetal force of

$$F_\mathrm{c} = \frac{mv^2}{r} = m\omega^2 r = 4\pi^2 m f^2 r\,,$$

where we express the frequency as $f = \frac{421\,\mathrm{rpm}}{60\,\mathrm{s}} \doteq 7.02\,\mathrm{Hz}$. This centripetal force (in the ground frame of reference) is the sum of the other forces, i.e. the force of gravity and the force imparted on the drum by the washing machine. We can express the latter as a vector $\mathbf{F} = \mathbf{F}_\mathrm{c} - \mathbf{F}_g$. It follows from the law of action-reaction that the same force acts on the washing machine. Its direction changes, sometimes pushing the washing machine to the ground, sometimes lifting it. Just when the laundry is above the drum's axis of rotation, the reaction to the centripetal force is directed upwards. In order for the washing machine to bounce up from the ground, this force has to be larger than its weight. From this we get the minimum weight of the washing machine as

$$M = m\left(\frac{4\pi^2 f^2 r}{g} - 1\right) \doteq 30.5\,\mathrm{kg}\,.$$

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Problem M.4 ... negative Moon 5 points

For the purposes of this problem, we can approximate the Earth as a perfect sphere with density $\rho_\mathrm{E} = 5.52\,\mathrm{g{\cdot}cm^{-3}}$. The same applies to the Moon, but its density is $\rho_\mathrm{M} = 3.34\,\mathrm{g{\cdot}cm^{-3}}$. We want to place electric charge uniformly in the whole volume of each body. What should be the value of the charge density (the same for both bodies) if we want the total interaction force between the two bodies to be zero?

Jáchym was walking on the street at night.

To meet the condition from the problem statement, the size of the attractive gravitational force has to be the same as the size of the repulsive electrostatic force. Marking the distance between the centres of the bodies as r, we get

$$\frac{G m_\mathrm{E} m_\mathrm{M}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_\mathrm{E} q_\mathrm{M}}{r^2}\,.$$

The mass can be calculated as the product of the density and volume, i.e. $m = \rho V$. If we mark the charge density ρ_c, a similar relation $q = \rho_c V$ applies for the charge. Substituting it to the previous equation we get

$$G\rho_E V_E \rho_M V_M = \frac{1}{4\pi\varepsilon_0}\rho_c V_E \rho_c V_M \quad \Rightarrow \quad \rho_c = \sqrt{4\pi\varepsilon_0 G \rho_E \rho_M} \doteq 3.70 \cdot 10^{-7}\,\mathrm{C{\cdot}m^{-3}}\,.$$

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Problem M.5 ... relaxing at a train station 5 points

Mišo is sitting in a house next to a railway, looking out for his favourite train. He only sees part of the tracks through the window, so he would like to know how much time he has until the train disappears from his view. He measured that the window is $c = 1.5\,\mathrm{m}$ wide and it's $p = 10\,\mathrm{m}$ far from the tracks. The chair he is sitting on is right in front of the center of the window, $l = 2.0\,\mathrm{m}$ far from it. Mišo also knows that the train is $d = 100\,\mathrm{m}$ long and its braking distance is $s = 150\,\mathrm{m}$. How much time does he have to watch the train (i.e. from the moment the locomotive appears to the moment the end of the train disappears from his view), if the train starts to evenly decelerate from a velocity $v = 20\,\mathrm{m{\cdot}s^{-1}}$ right at the time when Mišo first spots it? Neglect the width of the train. *Verča went on an expedition.*

Let us begin by finding the deceleration a of the train. We know the braking distance s of the train and its initial velocity v. Therefore we find the deceleration from the relations for uniformly accelerated motion as

$$s = \frac{1}{2}at^2 = \frac{v^2}{2a} \quad \Rightarrow \quad a = \frac{v^2}{2s}\,.$$

Now we calculate the distance travelled by the train while Mišo can watch it. We will need the length of the rails that are visible from the window; let us denote it b. The triangle formed by Mišo's chair and sides of the window is similar to the one formed by the chair and the furthest points of the rail that is visible from the window. Therefore

$$\frac{c}{b} = \frac{l}{l+p} \quad \Rightarrow \quad b = \left(1 + \frac{p}{l}\right)c\,,$$

holds. The distance the train will travel during Mišo's observation, equals the distance b and the length d of the train. Since we already know the deceleration of the train and the distance that it should travel, we can find the desired time by solving the quadratic equation, which yields

$$b + d = vt - \frac{1}{2}at^2 \quad \Rightarrow \quad t = \frac{2s}{v}\left(1 \pm \sqrt{1 - \left(\left(1 + \frac{p}{l}\right)c + d\right)\frac{1}{s}}\right)\,.$$

Plugging in the numerical values we get two results $t_1 \doteq 22.8\,\mathrm{s}$ and $t_2 \doteq 7.16\,\mathrm{s}$. The desired time is t_2, while t_1 marks the time, after which the train would get into the same position, if it moved backwards with the acceleration $-a$ after stopping.

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Problem M.6 ... the importance of having passengers 5 points

A frontal collision occurred between two cars. They both were the same models with the same manufacturer, each weighing $1\,200\,\mathrm{kg}$*, and both were moving at* $90\,\mathrm{km\cdot h^{-1}}$ *before the impact. One car was occupied only by a driver weighing* $80\,\mathrm{kg}$*, while the other one was occupied by more people (with total weight* $200\,\mathrm{kg}$*). The effects of the impact were mitigated by deformation of the whole engine spaces of the cars and the impact took* $80.0\,\mathrm{ms}$*. Calculate the average decelaration exerted on the crew of the lighter car in the units of the standard acceleration due to gravity* $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Jindra travelled with Dano by car.

In the reference frame connected to the ground, both cars move with the velocity $v = 90\,\mathrm{km\cdot h^{-1}}$, which equals $25\,\mathrm{m\cdot s^{-1}}$, against each other. The masses are different, therefore the centre of mass of the system moves as well. For simplicity, we move into the inertial reference frame connected with the centre of mass (centre of mass reference frame) of the two-car system. The acceleration does not change during transformations between inertial reference frames. Let $m_1 = 1\,280\,\mathrm{kg}$ denote the mass of the car with the driver only and $m_2 = 1\,400\,\mathrm{kg}$ the mass of a car with more passengers. In the centre of mass reference frame the first car moves with the velocity

$$v_1 = v\frac{2m_2}{m_1 + m_2}$$

while the second one with

$$v_2 = v\frac{2m_1}{m_1 + m_2}\,.$$

After the collision, both cars are at rest in this reference frame. The collision took $t = 80.0\,\mathrm{ms}$, therefore the average deceleration affecting the passengers of the lighter car can be calculated as

$$a_1 = \frac{v_1}{t} = \frac{v}{t}\frac{2m_2}{m_1 + m_2}\,.$$

Plugging in the numerical values we get $a_1 = 326\,\mathrm{m\cdot s^{-2}} = 33.3\,\mathrm{G}$ (G is the unit used in aerospace – ratio between the acceleration and the gravitational acceleration). Just for your information, the ratio between accelerations of the heavier and the lighter car is $a_1/a_2 = m_2/m_1 \doteq 1.09$.

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Problem M.7 ... a football problem 5 points

Jarda was watching the football Champions League and wondering how accurate the passes must be. Imagine a player passing the ball to his teammate. The teammate is running directly away from the passing player with a velocity $u = 7.0\,\mathrm{m\cdot s^{-1}}$*. At the time of the pass (when the passing player kicks the ball), he's standing on the offside line, which is at a distance* $L = 20\,\mathrm{m}$ *from the passing player. The passing player decides to kick the ball at an elevation angle* $\alpha = 30^\circ$ *with respect to the ground directly towards the running teammate. Find the velocity of the ball such that it falls just at the feet of the running player.*

Jarda watches football instead of attending lectures.

In order for the running player to get the ball exactly to his feet, he has to be in the place where the ball falls on the ground at the same time as the ball. Let's mark the time that the ball spends in the air as t. The initial vertical velocity is $v_y = v_0 \sin\alpha$ where v_0 is the initial velocity of the ball. The ball stops rising when the vertical velocity drops to zero, which happens in time $t_1 = \frac{v_y}{g}$. It will take the same amount of time for the ball to fall, therefore it spends a total of $t = \frac{2v_y}{g}$ in the air. During this time it flies a horizontal distance of

$$x = v_x t = v_0 \cos\alpha \frac{2v_0 \sin\alpha}{g} = \frac{{v_0}^2 \sin 2\alpha}{g}\,.$$

The running player has to be at this distance from the passer at the time when the ball falls. He had to cover the distance $s = ut = x - L$ during the time t. When substituting t, we get from this and the previous equation a quadratic equation for the initial velocity

$$v_0^2 \sin 2\alpha - 2uv_0 \sin\alpha - Lg = 0\,.$$

The solution is

$$v_0 = \frac{u \sin\alpha + \sqrt{u^2 \sin^2\alpha + Lg \sin 2\alpha}}{\sin 2\alpha} \doteq 19.6\,\mathrm{m\cdot s^{-1}}\,,$$

where we chose the positive root, as the negative one has no physical meaning.

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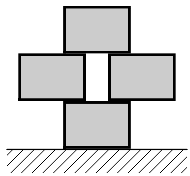
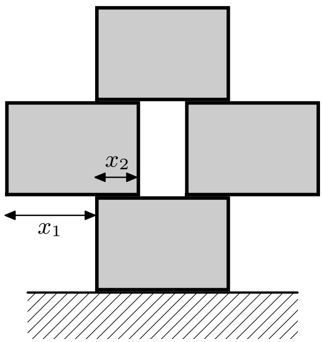
Problem M.8 ... a wall

5 points



We have four identical blocks (cuboids) with edge lengths $a \times b \times b$, where $a = 29.7\,\mathrm{cm}$ and $b = 21.0\,\mathrm{cm}$. We put these blocks on top of each other as shown in the figure. What is the maximum width of the gap between the two middle blocks? *Jáchym wanted to build a wall.*

If the brick was about to fall, it would turn around the edge of the lowest brick. This edge divides the middle brick into two regions $x_1 + x_2 = a$ as you can see in the picture.



The total torque with respect to this axis must be zero. The left part has mass

$$m_1 = \frac{x_1}{a} m\,.$$

The centre of mass of this part of the brick is $x_1/2$ from the edge of the bottom brick. The torque caused by the force of gravity is $x_1 m_1 g/2$.

Similarly, the right part of the brick experiences torque $x_2 m_2 g/2$ and the torque caused by the top brick, too. Its weight is equally spread across the two bricks. This force acts in the upper right edge of the middle brick (imagine that the top brick rotates by a small angle), which means that the perpendicular distance is x_2 and the torque is $x_2 mg/2$.

The torques must be equal

$$\frac{x_1 m_1 g}{2} = \frac{x_2 m_2 g}{2} + \frac{x_2 m g}{2}\,,$$
$$x_1^2 = x_2^2 + a x_2\,.$$

We substitute $x_1 = a - x_2$ and get

$$x_2 = \frac{1}{3} a\,.$$

The maximum width of the gap is

$$x = a - 2x_2 = \frac{1}{3} a \doteq 9.9\,\mathrm{cm}\,.$$

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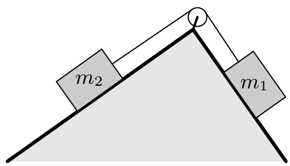
Problem M.9 ... pulleys on an inclined block — 5 points



Consider a prism whose bases are right triangles. A horizontal view of the prism is shown in the figure (the bases are vertical). On its two perpendicular side faces, there are two blocks with masses $m_1 = 15\,\mathrm{kg}$ and $m_2 = 10\,\mathrm{kg}$ connected by a rope through a pulley. What must be the angle between the side with the block with the mass m_1 and the horizontal plane if we want to keep the blocks at rest, assuming there is no friction in the system?

Besides pulleys, Lego likes problems that are similar.

Three different forces are acting on both blocks:

- gravitational force, acting downwards with magnitude $m_i g$,
- normal force from the face of the prism – this one prevents the blocks from "falling through" by cancelling out the component of the gravitational force, which is perpendicular to the face of the prism – therefore only the component parallel to the wall remains,
- the pull force of the rope, which pulls both blocks upwards, parallel to the wall.

If we want the blocks to remain in rest, then the force, which pulls the rope, must be equal to the magnitude of the component of the gravitational force, which is parallel to the wall. Furthermore, consider that it has the same magnitude for both blocks (otherwise the net force acting on the rope would be non-zero, which will cause the rope and therefore the blocks to accelerate). We conclude that the component of gravitational force parallel to the wall must be equal for both blocks, yielding

$$F_{\mathrm{p},1} = F_{\mathrm{p},2}\,,$$
$$m_1 g \sin\varphi = m_2 g \sin\left(\frac{\pi}{2} - \varphi\right) = m_2 g \cos\varphi\,,$$
$$\varphi = \arctan\frac{m_2}{m_1} \doteq 34^\circ\,.$$

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Problem M.10 ... A

5 points

The letter A consists of two long rods, each with a length $4l$, that are connected by a hinge at the top and by a rope with a length l at their midpoints. If each of these rods has a diameter $r = l/64$ and is made of stiff metal with a density $\rho = 6\,850\,\mathrm{kg\cdot m^{-3}}$, what is the tallest letter A we can build? The rope has the same radius as the rods, but it is made of a material with negligible density, which can handle tensile stress up to $\sigma = 4.50\,\mathrm{MPa}$. Neglect friction.

Jáchym wanted the first problem (at least alphabetically) to be his.

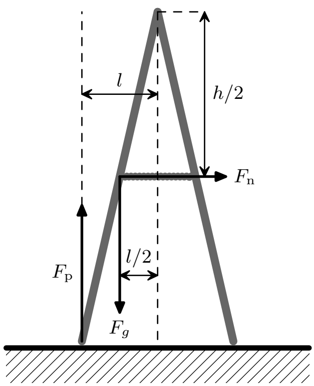
The length of the rod is $4l$, so its mass is

$$m = 4\pi\rho r^2 l\,.$$

The tensile stress on the rope is σ, so the force it imparts on each rod is

$$F_\mathrm{s} = \pi r^2 \sigma\,.$$

Each rod also experiences the reactive force from the ground, the magnitude of which is equal to the force of gravity $F_g = mg$ acting in the centre of mass. The last force is acting at the hinge between the rods, but we won't need its magnitude.



The total torque acting on the rods must be zero to keep the system in balance. Determining the torques relative to the point of the hinge, the moment arm of the reaction force from the ground is l, the momentum arm of the tension force is $h/2$ and the arm of the gravitational force is $l/2$, where h is the height of the letter for which we can write

$$h = \sqrt{(4l)^2 - l^2} = \sqrt{15}l\,.$$

From the condition of zero torque, we obtain

$$0 = lF_g - \frac{l}{2}F_g - \frac{h}{2}F_\mathrm{s}\,,$$
$$0 = \frac{4\pi\rho r^2 l^2 g}{2} - \frac{\sqrt{15}\pi r^2 l\sigma}{2}\,,$$
$$l = \frac{\sqrt{15}\sigma}{4\rho g}\,.$$

Notice, that the bigger the tension, the bigger l and the bigger h. To determine the maximum height of the letter, we use the maximum tensile stress and obtain

$$h = \sqrt{15}l = \frac{15\sigma}{4\rho g} \doteq 251\,\mathrm{m}\,.$$

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Problem M.11 ... falling coins 5 points

We have many identical coins (discs with radii $r = 1\,\mathrm{cm}$) sliding down an inclined plane. The inclination angle of the plane is $\alpha = 45^\circ$ and it is $l = 50\,\mathrm{cm}$ long. At the bottom of the slope, there is a $d = 65\,\mathrm{cm}$ long horizontal plane. Each coin is released from rest at the top of the slope. What is the smallest number of coins (including the first coin) that we have to slide down the plane to ensure that the first coin will fall off the edge of the horizontal plane? Assume perfectly elastic collisions and a constant friction coefficient $f = 0.5$ between the coins and the surface. *Kiko knows his way around money.*

When a coin slides along the inclined plane, a force perpendicular to the plane acts upon it

$$F = mg\sin\alpha - fmg\cos\alpha\,.$$

The resultant acceleration of the coin is

$$a_1 = (\sin\alpha - f\cos\alpha)\,g\,.$$

That is an uniformly accelerated motion. The final velocity after sliding down the plane will be

$$v = \sqrt{2a_1 l}\,.$$

After that, another uniformly accelerated motion follows, this time slowing down due to friction

$$a_2 = fg\,.$$

The center of the first coin will then reach the distance of

$$x = \frac{v^2}{2a_2} = \frac{a_1}{a_2} l = \left(f^{-1} \sin\alpha - \cos\alpha\right) l \doteq 35.4\,\mathrm{cm}\,.$$

Thanks to equal mass of all the coins and the elasticity of the collisions the moving coin will always stop during the collision and transfer all its momentum to the next coin (as it is the case with Newton's pendulum). Thanks to this, the first coin move the distance equal to its diameter with each collision. This result could be calculated, but it is not neccesary since in the end, each collision is essentialy equivalent to teleporting the first coin forward by the distance equal to its diameter, while there are no losses of energy through friction during this "teleportation". As a result, with each successive coin, the same amount of energy must be lost and there is one more coin (and the whole system moves the distance of one diameter of the coin).

This way, we have created a pattern of holes and coins with the same width. The problem would be vastly more complicated if the coins started to accumulate at the spot where the incluned plane meets the horizontal plane. Luckily, that will not happen since the pattern is always symmetrical about the point where the center of the first coin originally stopped. And this point is sufficiently far from the center of the horizontal plane.

For the first coin to fall of the edge, its center must go beyond it. So the total number of coins n is given by the condition

$$2\,(n-1)\,r + x > d\,.$$

Which gives

$$n > \frac{d-x}{2r} + 1\,.$$

Numerically, we obtain $n = 16$.

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Problem M.12 ... leaping coin
5 points

Kiko placed a thin coin with mass $m = 4\,\mathrm{g}$ on the opening of a bottle of (still) iced tea, which was cooled down to $5\,^\circ\mathrm{C}$. The coin started rattling after a while. Determine how many times the coin lifted from the opening if the ambient temperature of the surrounding air was $25\,^\circ\mathrm{C}$ and the radius of the coin was $r = 0.5\,\mathrm{cm}$. *Kiko was bored while drinking beer.*

After taking the bottle out of a fridge, the air inside it start to heat up and therefore expand. Coin will jump when inside pressure rises enough to overcome coin's weight thus making the inside and outside pressures equal $p_1 = p_\mathrm{a}$. While the bottle is sealed, the gas undergoes an isochoric process

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}\,.$$

Pressure difference required to lift the coin is

$$\Delta p = p_2 - p_\mathrm{a} = \frac{mg}{\pi r^2}\,.$$

Temperature inside the bottle after first jumps becomes

$$T_2 = p_2 \frac{T_1}{p_1} = \left(\frac{mg}{\pi r^2} + p_\mathrm{a}\right) \frac{T_1}{p_\mathrm{a}} = \left(\frac{mg}{\pi r^2 p_\mathrm{a}} + 1\right) T_1 \,.$$

After n^th jump the temperature inside the bottle becomes

$$T_n = \left(\frac{mg}{\pi r^2 p_\mathrm{a}} + 1\right)^n T_1 \,,$$

which is bound by temperature of surroundings at $25\,^\circ\mathrm{C}$. The number of jumps can be obtained from inequality

$$T_n < 25\,^\circ\mathrm{C} \,.$$

The maximum number of jumps can be computed either by logarithm or íteration. We need to remeber to use the Kelvin scale for temperature, finaly obtaining the result $n = 14$.

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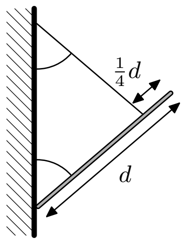
Problem M.13 ... climbing 5 points



A mountain climber with weight $m = 75\,\mathrm{kg}$ and height $d = 180\,\mathrm{cm}$ is climbing up a vertical wall. At some point, he is standing in such a way that the angle between his body and the wall is the same as the angle between the wall and the rope. The climber is connected to the rope via a full-body harness at the distance $\frac{3}{4}d$ from the point where his shoes are touching the wall. The current friction force between the mountaineer's shoes and the wall is exactly the threshold static friction force. Find the magnitude of the force acting on (stretching) the rope. The coefficient of static friction between the wall and the shoes is $f = 0.40$. Assume that gravity acts on the mountain climber in his centre of mass, located at the midpoint of his body.

Danka was wondering about climbing.

Let's mark the angle mentioned in the problem statement as α, $k = 3/4$ and F is the force the rope exerts on him. Further, the climber is affected by the normal force from the wall F_n, the frictional force F_f and the gravitational force F_g. The horizontal components of these forces must be in balance, so

$$F_\mathrm{n} - F\sin\alpha = 0\,.$$

Similarly, the sum of the vertical components must be zero

$$F_\mathrm{f} + F\cos\alpha - F_g = 0\,.$$

Applying the condition of zero total torque relative to the point, where the climber touches the rope, we get

$$F_\mathrm{f}kd\sin\alpha - F_\mathrm{n}kd\cos\alpha - F_g\left(k - \frac{1}{2}\right)d\sin\alpha = 0\,.$$

The magnitudes of frictional and gravitational forces are the last two needed relations, written as

$$F_\mathrm{f} = fF_\mathrm{n}\,,$$
$$F_g = mg\,.$$

Now, we have five equations for five variables (four forces and one angle). The first step in the solution is to substitue in the frictional force as above. Next, from the first two equations, we express the required tension of the rope

$$F = \frac{F_\mathrm{n}}{\sin\alpha} = \frac{F_g - fF_\mathrm{n}}{\cos\alpha} \quad\Rightarrow\quad \tan\alpha = \frac{F_\mathrm{n}}{F_g - fF_\mathrm{n}}\,.$$

We got rid of F by this formula and we got tangent of an angle α. This can also be easily expressed from the torques equation, from which we get

$$\tan\alpha = \frac{2F_\mathrm{n}k}{2fF_\mathrm{n}k - F_g\left(2k-1\right)}\,.$$

If we put both formulas for $\tan\alpha$ together, we finally get the relation between normal and gravitational force

$$F_\mathrm{n} = \frac{F_g\left(4k-1\right)}{4fk}\,.$$

Putting this into the previous equation we get

$$\tan\alpha = \frac{4k-1}{f} \quad \Rightarrow \quad \sin\alpha = \frac{4k-1}{\sqrt{f^2 + (4k-1)^2}}\,,$$

where we used the mathematical identity

$$\sin\arctan x = \frac{x}{\sqrt{1+x^2}}\,.$$

Now we just plug in one of the previous expressions for F and we get the result

$$F = \frac{F_\mathrm{n}}{\sin\alpha} = \frac{F_g\,(4k-1)}{4fk}\frac{\sqrt{f^2+(4k-1)^2}}{4k-1} = \frac{mg}{4k}\sqrt{1+\left(\frac{4k-1}{f}\right)^2} \doteq 1\,250\,\mathrm{N}\,.$$

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Problem M.14 ... an impact into a wall — 5 points

Originally, Jindra wanted to let a rolling cylinder impact a wall, but he realised that he cannot calculate how the impact would progress. Therefore, he took a homogeneous cylinder with a mass $m = 12\,\mathrm{kg}$, length $l = 50\,\mathrm{cm}$ and radius $r = 10\,\mathrm{cm}$, let it spin around its axis with an angular velocity $\omega = 50\,\mathrm{rad{\cdot}s^{-1}}$ and placed it on the floor next to the wall in such a way that it was touching both the floor and the wall and slipping occured at both of these points of contact and the cylinder was rolling into the wall (if the wall did not exist, it would roll in its direction). The coefficient of friction between the cylinder and the floor is $\mu_1 = 0.38$, and between the cylinder and the wall it is $\mu_2 = 0.57$. Find the magnitude of the friction force between the cylinder and the wall. *Jindra was demolishing a wall.*

When the cylinder touches the wall, several forces act on it. The force of gravity $\mathbf{F}_g$ acts downwards, the friction force $\mathbf{T}_1$ from the floor decelerates the rotation and pushes the cylinder towards the wall, the reaction force $\mathbf{N}_2$ from the wall counteracts the effects of the friction force from the floor, the friction force $\mathbf{T}_2$ from the wall acts upwards and decelerates the rotation, and the reaction force $\mathbf{N}_1$ from the floor counteracts the effects of the force of gravity and the friction force from the wall. Since the cylinder skids, the magnitudes of the friction forces are maximum possible (i.e. the coefficient of friction multiplied by the magnitude of the normal force). Balancing the forces in both directions

$$T_1 = N_2\,,$$
$$T_2 = mg - N_1\,.$$

We also have the defining relations for the friction force $T_i = \mu_i N_i$. That is a system of linear equations with four variables (N_1, N_2, T_1, T_2) and we want to find T_2. First, we substitute the appropriate $T_i\mu_i^{-1}$ for N_i. Then, we express the force from the first equation

$$T_1 = \mu_2^{-1} T_2\,,$$

and substitute it to the second one, and then we obtain

$$T_2 = mg - (\mu_1\mu_2)^{-1} T_2 ,$$
$$T_2 = \frac{mg}{(\mu_1\mu_2)^{-1} + 1} .$$

After substituting the numbers from the problem statement, we get $T_2 \doteq 21.0\,\mathrm{N}$.

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Problem M.15 ... non-quantum entanglement 5 points

Assume that we have a point from which a pair of photons originates. Polarisations of these photons are perpendicular to each other. As the name of this problem suggests, we do not claim that they are in superposition – both photons have given polarisations, but we do not know which. Furthermore, for each photon, we place a polariser perpendicularly to its trajectory. Assuming that the planes of polarisation of the polarisers are parallel to each other, what is the probability that both photons from a given pair pass through the polarisers?

Lego was preparing his lecture for a physics camp.

Let's mark the angle between the polarization plane of one photon and the polarization plane of the polarizers φ. The probability that this photon will pass through the polarizer is $p_1 = \cos^2\varphi$. The equally defined angle for the second photon is $\pi/2 - \varphi$ and the probability that it flies through the polarizer is $p_{12} = p_1 p_2 = \cos^2\varphi \sin^2\varphi = 1/4 \sin^2(2\varphi)$. So the probability that both photons fly through, is

$$p_{12} = p_1 p_2 = \cos^2\varphi \sin^2\varphi = 1/4 \sin^2(2\varphi) .$$

The angle φ is random and it is needed to define the expected value

$$\overline{p}_{12} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} p_{12}(\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \sin^2(2\varphi)\,\mathrm{d}\varphi = \frac{1}{2\pi}\frac{\pi}{4} = \frac{1}{8} ,$$

where it is enough for the integration to know that the integral of $\sin^2$ is through an integer multiple of its half-period, which is always equal to half of the interval. We could just as well write the average value through the whole 2π, but we could also do it through $\pi/4$. Let us now note that when (for instance) positronium decays into two photons with perpendicular polarisation, and if we put the polarisers with parallel polarisation planes between them, we wouldn't observe any cases where both photons pass. This means that photon polarisation is not determined in the moment of decay, but only in the process of measurement.

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Problem M.16 ... water planet

5 points

Consider a non-rotating planet with a radius $R = 2\,000\,\mathrm{km}$ *formed only by liquid water and an atmosphere. The atmospheric pressure on the surface is* $p_\mathrm{a} = 10\,\mathrm{kPa}$*. What is the pressure in its center? Assume that water is incompressible.*

Honza was wondering what could exist in the universe.

For every part of the planet hydrostatic equilibrium condition holds

$$\nabla p = \rho \mathbf{a}_\mathrm{g} \,,$$

where p is pressure, ρ is density of the water and $\mathbf{a}_\mathrm{g}$ is gravitational acceleration. Due to symmetry we can rewrite the condition in spherical coordinates as

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \rho a_\mathrm{g} \,.$$

As per well known consequence of Gauss's theorem, gravitational acceleration at distance r from the center of a spherically symmetrical mass distribution is exerted only by the mass enveloped in a sphere of diameter r of total mass

$$m = \frac{4}{3}\pi r^3 \rho$$

that acts like a point mass, giving gravitational acceleration at r from the center

$$a_\mathrm{g} = -\frac{Gm}{r^2} = -\frac{4\pi G r \rho}{3} \,,$$

where the sign means that the force acts in direction of decreasing r. Substituting into previous equations we get

$$\mathrm{d}p = -\frac{4\pi G \rho^2}{3} r \,\mathrm{d}r \,.$$

Integrating both sides between boundaries at the surface and the center we get central pressure

$$\int_{p_0}^{p_\mathrm{a}} \mathrm{d}p = -\int_0^R \frac{4\pi G \rho^2}{3} r \,\mathrm{d}r \,,$$

$$p_\mathrm{a} - p_0 = -\frac{2\pi G \rho^2}{3} R^2 \,,$$

$$p_0 = p_\mathrm{a} + \frac{2\pi G \rho^2}{3} R^2 \doteq 557\,\mathrm{MPa} \,.$$

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Problem M.17 ... slot car racing

5 points

When Jarda was a child, he didn't organise FYKOS, so he had a lot of free time. Once, he got a present from his parents – a slot car track with a shape of a very tall vertical helix, together with slot cars, each with a mass $m = 80\,\mathrm{g}$. The radius of the helix is $r = 10\,\mathrm{cm}$ and a car moves $h = 8\,\mathrm{cm}$ down per one turn. When the car is moving down, it slides on the outer edge of the track where the friction coefficient is $\mu = 0.3$. Jarda launched the first car at a time t_0 (with zero initial velocity) and the second car from the same point five seconds later (also with zero initial velocity). At each point in time afterwards, the first car will be some number of turns ahead of the second car. What is the maximum value of this difference in the travelled number of turns? Assume that each car is a point mass. *Jarda was reminiscing about his childhood.*

We can untwist the helix car track to a usual straight slope. The car will be accelerated by the component of gravitational force parallel to the slope

$$F_g^{\mathrm{t}} = mg\sin\alpha\,,$$

where α is the angle between the slope and a horizontal plane. The car is rolling on wheels with negligible resistance from the surface underneath. By untwisting one story of helix we get a right triangle with sides h and $2\pi r$, therefore we get angle

$$\sin\alpha = \frac{h}{\sqrt{h^2 + (2\pi r)^2}}\,.$$

There is, however, aditional friction force. The car is kept on a circular trajectory of radius r by reaction force from outer side of the track

$$F_{\mathrm{d}} = \frac{m v_{\mathrm{h}}^2}{r}\,,$$

where v_{h} is the horizontal projection of velocity $v_{\mathrm{h}} = v\cos\alpha$. For cosine of angle α we have

$$\cos\alpha = \frac{2\pi r}{\sqrt{h^2 + (2\pi r)^2}}\,.$$

This force is causing friction force from the track side

$$F_{\mathrm{f}} = \mu F_{\mathrm{d}} = \frac{\mu m v_{\mathrm{h}}^2}{r} = \frac{\mu m v^2}{r}\cos^2\alpha\,.$$

Now we can complete the equation of motion

$$F = F_g^{\mathrm{t}} - F_{\mathrm{f}} \quad \Rightarrow \quad \dot{v} = g\sin\alpha - \frac{\mu v^2}{r}\cos^2\alpha\,.$$

This differential equation can be solved, but it is enough to understand that there is certain terminal velocity of cars. We can obtain this velocity by the condition $a = 0$ as

$$v_{\mathrm{t}} = \sqrt{\frac{g r\sin\alpha}{\mu\cos^2\alpha}}\,.$$

This velocity is obtained in infinite time, until then the cars are accelerating more slowly as time goes by. The distance between cars will be increasing up to a maximum value of $\Delta s = v_\mathrm{t} t$, where $t = 5\,\mathrm{s}$ is starting time difference. For height difference we get

$$\Delta h = \Delta s \sin\alpha\,.$$

and for the number of turns one car is preceeding the other as time goes to infinity

$$\frac{\Delta h}{h} = \frac{t}{2\pi}\sqrt{\frac{gh}{\mu r\sqrt{h^2 + (2\pi r)^2}}} \doteq 5.1\,.$$

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Problem E.1 ... the anti-diet of the Earth 5 points

How would the mass of the Earth increase if its mean radius was increased by $\Delta r = 1.0\,\mathrm{m}$ by adding material with the same density as Earth's mean density? Find the mass of this added material. *Karel was thinking about meteorite impacts.*

First, we determine the density of the Earth. We could simply find this online, but let's calculate it from the quantities we know. We will assume that the Earth is a sphere with radius of $R_\oplus \doteq 6.378 \cdot 10^6\,\mathrm{m}$. In reality, this is its radius on the equator, but the difference is relatively small. The mass of the Earth is $M_\oplus = 5.974 \cdot 10^{24}\,\mathrm{kg}$. Density can then be determined as

$$\rho_\oplus = \frac{M_\oplus}{V_\oplus} = \frac{3M_\oplus}{4\pi R_\oplus^3} \doteq 5\,497\,\mathrm{kg{\cdot}m^{-3}} \doteq 5\,500\,\mathrm{kg{\cdot}m^{-3}}\,.$$

If we compare it with the density found online, for example, on Wikipedia, where $5\,515\,\mathrm{kg{\cdot}m^{-3}}$ is given, our estimation is enough for a result valid to two significant digits.

The whole solution can be very simple if we realize that the radius of the Earth will increase only by a small amount. Therefore, the change to the surface area $S_\oplus$ will be negligible. The change in volume of the Earth is then the product of the Earth's surface area and the change in height

$$\Delta V = S_\oplus \Delta r = 4\pi R_\oplus^2 \Delta r \doteq 5.11 \cdot 10^{14}\,\mathrm{m}^3\,.$$

The mass will change by

$$\Delta m = \rho_\oplus \Delta V = \frac{3M_\oplus}{4\pi R_\oplus^3} 4\pi R_\oplus^2 \Delta r = \frac{3M_\oplus}{R_\oplus}\Delta r \doteq 2.8 \cdot 10^{18}\,\mathrm{kg}\,.$$

The whole mass of the Earth will increase by $2.8 \cdot 10^{18}\,\mathrm{kg}$, what is $0.000\,047\,\%$ of the original mass.

Alternately, we can count the mass of extended Earth and subtract the original mass from it. Due to small increase in radius, this should be a negligible difference. Let's make sure of that

$$\Delta m' = \rho_\oplus \Delta V' = \frac{3M_\oplus}{4\pi R_\oplus^3}\left(\frac{4}{3}\pi\left(R_\oplus + \Delta r\right)^3 - \frac{4}{3}\pi R_\oplus^3\right)$$
$$= \frac{M_\oplus}{R_\oplus^3}\left(3R_\oplus^2\Delta r + 3R_\oplus \Delta r^2 + \Delta r^3\right)\,.$$

We can see that the result has three terms. The first term is the result that we got in the approximate calculation and the other members are really negligible and after rounding we get the same result. We can use this approximation as long as the increase in radius is significantly smaller than the original radius of the Earth.

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Problem E.2 ... Diomede Islands 5 points

Two small islands are located only 3.7 km apart in the Bering strait. The west island belongs to Russia while the east one is part of the USA. The interesting thing about them is that the International Date Line passes just between them. That is why they are sometimes called Tomorrow Island and Yesterday Island). Once in the winter, when the strait was frozen, one traveller *walked from the American island to the Russian island with an average pace of* $4\,\mathrm{km\cdot h^{-1}}$ *(don't try this, it's illegal). However, his GPS showed him a very bizzare average speed since it calculated both the time of departure and the time of arrival using the local time zones. What was this average speed?*

The Russian island is in the time zone +12, the American one in the time zone -9. Be careful – the average speed has a sign, too. *Matěj would like to travel, but can't do it now.*

The trick is to realize the direction in which the Earth rotates. For example, when it is 13 February 12:01pm on the Greenwich meridian, on the Russian island it is 00:01 (12:01am) on the 14$^{\mathrm{th}}$ and on the American one it is 3:01am, February 13th. Therefore we have to add a false $12\,\mathrm{h} - (-9\,\mathrm{h}) = 21\,\mathrm{h}$.

The journey actually took him $\frac{3.7\,\mathrm{km}}{4\,\mathrm{km\cdot h^{-1}}} = 0.925\,\mathrm{h}$, and the false average speed is therefore

$$\frac{3.7\,\mathrm{km}}{21.925\,\mathrm{h}} = 0.168\,8\,\mathrm{km\cdot h^{-1}} = 4.69\,\mathrm{cm\cdot s^{-1}}\,.$$

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Problem E.3 ... water on Mars 5 points

Mars doesn't have much water. Imagine that we would like to create an ocean on Mars and transport all the water from the Pacific Ocean there, which would create a "new Pacific Ocean" covering almost the entire Mars. What would be the difference in average depth between these two oceans? The answer should be a negative number if the new ocean is shallower. Assume the Pacific Ocean covers a third of the Earth's surface and contains $7.1\cdot 10^{8}\,\mathrm{km^3}$ *of water. The radius of Mars is* $3\,390\,\mathrm{km}$. *For simplicity, neglect the rotation of Mars.* *Matěj was thirsty.*

The surface area of the Earth is $S_\mathrm{E} = 4\pi R_\mathrm{Z}^2$. Since the depth of the ocean is very small compared to the radius of the Earth, we can calculate the average depth as

$$h_0 = \frac{V}{\frac{1}{3}S_\mathrm{E}} = \frac{3V}{4\pi R_\mathrm{E}^2}\,,$$

where V is the volume of the Pacific and $R_\mathrm{E} = 6\,378\,\mathrm{km}$. The average depth of the Pacific ocean on Mars would be

$$h_1 = \frac{V}{S_\mathrm{M}} = \frac{V}{4\pi R_\mathrm{M}^2}\,,$$

where $R_\mathrm{M} = 3\,390\,\mathrm{km}$ is the radius of Mars. The difference in depths is

$$h_1 - h_0 = \frac{V}{4\pi}\left(\frac{1}{R_M^2} - \frac{3}{R_\mathrm{E}^2}\right) \doteq 750\,\mathrm{m}\,.$$

The Pacific would be almost one kilometer deeper.

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Problem E.4 ... rolling in the Neanderthal way 5 points

Ancient civilisations invented the wheel long ago. Determine the steepness of a hill needed for a regular 42-gon to start rolling down the hill on its own.

Kiko was checking out a boneshaker bike in a museum of technology.

In order for the body to tip over, the position of its center of gravity must be directly above the vertex around which is the body turning (actually, it is a bit more to the right but we are solving the critical point). For the angle between the side of the n-gon and the line passing through the center of gravity, and the corresponding vertex of the side we get

$$\varphi = \frac{\pi - \frac{2\pi}{n}}{2}\,.$$

To make the body tip over, the angle must be $\pi/2$. The steepness of the hill adds to this, so it must be

$$\alpha = \pi/2 - \varphi = \frac{\pi}{n}\,.$$

For 42-gon we get approximately $4.3\,^\circ$.

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Problem E.5 ... dry 5 points

The relative humidity in Danka's room is an unpleasant $\Phi_{r,1} = 20\,\%$. How much water does Danka need to evaporate to increase the humidity to $\Phi_{r,2} = 50\,\%$? The room contains $V = 15\,\mathrm{m}^3$ of air with temperature $25°\mathrm{C}$ and this air contains $\Phi_1 = 4.6\,\mathrm{g{\cdot}m^{-3}}$ of vaporized water.

Danka's room was too dry.

Absolute humidity of air is defined as

$$\Phi = \frac{m}{V}\,,$$

where m is the mass of the water in the air and V is the total volume of the air. Relative humidity is then

$$\Phi_\mathrm{r} = \frac{\Phi}{\Phi_m}\,,$$

where Φ_m is the mass of water vapor per unit of volume, at which the air is saturated with water at a given temperature. From the actual situation, we determine the amount of water in $1\,\mathrm{m}^3$ at saturation Φ_m, as

$$\Phi_m = \Phi_1/\Phi_{\mathrm{r},1}\,.$$

The difference of the mass of the water in the air is expressed from the first equation as

$$\Delta m = m_2 - m_1 = V\left(\Phi_2 - \Phi_1\right) = V\Phi_m\left(\Phi_{\mathrm{r},2} - \Phi_{\mathrm{r},1}\right)\,.$$

Using Φ_m, we finally get

$$\Delta m = V\Phi_1\left(\frac{\Phi_{\mathrm{r},2}}{\Phi_{\mathrm{r},1}} - 1\right) \doteq 104\,\mathrm{g}\,.$$

Danka needs to evaporate approximately $104\,\mathrm{g}$ of water.

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Problem E.6 ... hanging clothes 5 points

Verča hangs towels on the clothes rack in such a way that she throws them over a rod with a square cross-section. One day, she noticed that part of a towel, which does not overlap with the other side, dries faster. Therefore, she decided to hang the towels in such a way that the overlaping part is as small as possible, but the towel still does not slide down. What is the highest ratio of the two hanging parts (the longer part to the shorter part) for Verča's towel such that it remains secured from falling? The coefficient of friction between the rod and the towel is $f = 0.2$ and the mass of the towel is $m = 1\,\mathrm{kg}$.

Verča is looking for physics even in doing household chores.

Let's mark the mass of the longer part as m_1 and the shorter part as m_2, assuming that the mass of the towel is distributed equally. One part will be subjected to the force of gravity $F_{g,1} = m_1 g$, the other to $F_{g,2} = m_2 g$. A force of friction $F_\mathrm{f} = mgf$ will act against the larger of the forces of gravity ($F_{g,1}$) and hold the towel on the rod. We get the equation

$$F_{g,2} + F_\mathrm{f} = F_{g,1}\,,$$

and substituting in the forces

$$m_2 g + mgf = m_1 g\,.$$

We also know that $m_1 + m_2 = m$; therefore after substituting for m we can express the mass ratio as

$$\frac{m_1}{m_2} = \frac{1+f}{1-f}\,.$$

The ratio of the masses is the same as the ratio of the lengths, therefore after substituting numbers we get the result $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{3}{2}$. We can notice that the only thing that affects the result is the coefficient of friction; it does not depend on the mass nor the gravitational acceleration.

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Problem E.7 ... gravitation 5 points

Matěj would love to walk on the Moon, but it's hard to get there. Instead, he used a little trick – he filmed himself jumping on the Earth and then played the video in slow motion. At what frame rate does he need to play the clip to make it look like jumping on the Moon? The camera has a frame rate of $60\,\mathrm{fps}$. Assume that the gravity on the Moon is six times smaller than the gravity on the Earth. *Matěj and film tricks.*

We will consider only the vertical component of the trajectory since the other components are not affected by the gravitational force.

For the movement in a homogenous gravitational field g we have

$$y(t) = -\frac{1}{2} g t^2 + v_0 t + y_0\,,$$

where v_0 is the (vertical) speed at the time $t = 0$ and y_0 is the position at the same time. Now we will transform from unprimed to primed variables representing the altered gravity and slower time. We want the resultant equation to be the same as original since the vertical position y is the same in both the original and the slowed down clip

$$y(t) = y'\left(t'\right) = -\frac{1}{2}g't'^2 + v_0't' + y_0' \,.$$

We also want $g' = g/6$. Comparing the two equations (the equation $y(t) = y'(t')$ must be satisfied for all possible values of v_0 and y_0) we get

$$t' = t\sqrt{6}\,, \quad v_0' = \frac{v_0}{\sqrt{6}}\,, \quad y_0' = y_0\,. \tag{1}$$

We can clearly see that the time must be slowed down by the factor of $\sqrt{6}$, that is while in the original clip 60 frames pass, in the slowed clip, it will only be $\frac{60}{\sqrt{6}} \doteq 24.5$ frames.

Althought that was a clever trick, it has one disadvantage. Matěj's speed during the take-off will, because of (1), appear to be $\sqrt{6}$-times lower. So it will seem Matěj can only jump with a very small speed.

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Problem E.8 ... rope on an inclined plane — 5 points

Consider an inclined plane with an angle $\varphi = 50^\circ$ between this plane and the horizontal plane. A rope with a linear density $\lambda = 1.5\,\mathrm{kg{\cdot}m^{-1}}$ is laid on it, along the plane's entire length. The height difference between the top and bottom point of the plane is $h = 1.7\,\mathrm{m}$. What is the force with which we have to hold the rope so it doesn't slip down, if there is zero friction between the rope and the plane? *Lego doesn't feel like coming up with origins anymore, sorry :D*

The mass of the rope is $m = \lambda h/\sin\varphi$. Considering that the angle is constant, we assume that all the mass is located in the center of mass and we will treat it as a point mass. Then, we know that the normal component of the force of gravity is counteracted by normal force from the plane and we only need to couunteract the component parallel with the plane. The magnitude of this component is $F = F_g \sin\varphi = g\lambda h \doteq 25\,\mathrm{N}$.

We can see that the force doesn't depend on the angle, but the only relevant information is the height difference.

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Problem X.1 ... massive crash 5 points

Two point masses $m_1 = 50.0\,\mathrm{kg}$ and $m_2 = 60.0\,\mathrm{kg}$ are $l = 6.00\,\mathrm{m}$ apart at rest. Suddenly, they start attracting each other with a constant force $F = 100\,\mathrm{N}$. What velocity will they collide with? *Verča was reminiscing about skating with her classmates.*

As the first step, let's find the positions of the two point masses at a time t. Suppose that the first point mass is at the origin of our coordinate system and the second point mass is at $x = l$ at the beginning. Because the force is constant, the masses accelerate uniformly. We can calculate each acceleration as the force divided by the mass $a_1 = F/m_1$ and $a_2 = F/m_2$. The positions of the two point masses at a time t are

$$x_1 = \frac{1}{2}a_1 t^2 = \frac{1}{2}\frac{F}{m_1}t^2\,,$$
$$x_2 = l - \frac{1}{2}a_2 t^2 = l - \frac{1}{2}\frac{F}{m_2}t^2\,.$$

They crash when $x_1 = x_2$

$$\frac{1}{2}\frac{F}{m_1}t^2 = l - \frac{1}{2}\frac{F}{m_2}t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2lm_1m_2}{F\left(m_1+m_2\right)}}\,.$$

Now, we have to calculate their velocities using $v = at = \frac{F}{m}t$. The velocity v of one mass relative to the other is simply the sum of the two velocities

$$v = v_1 + v_2 = Ft\left(\frac{1}{m_1}+\frac{1}{m_2}\right) = \sqrt{\frac{2lm_1m_2}{F\left(m_1+m_2\right)}}\frac{F\left(m_1+m_2\right)}{m_1m_2} = \sqrt{\frac{2Fl\left(m_1+m_2\right)}{m_1m_2}}\,.$$

The two point masses collide with velocity $v = 6.63\,\mathrm{m{\cdot}s^{-1}}$.

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Problem X.2 ... Pythagorean theorem not working 5 points

Matěj doesn't like winding paths and he prefers to walk in a straight line, especially when he walks exactly in one of the cardinal directions. He once wondered about the most beautiful way to travel from Prague to Brno. He could either first go straight east and then south, or first go straight south and then east (always making just one 90° turn on the way). What is the difference between the lengths of the first path and the second path? (If the first path is shorter, the answer should be a negative number.) In reality, the specific shape of the terrain would have much greater effect on the result than curvature of the Earth - therefore, consider the Earth to be a perfect sphere. The coordinates of Prague are $50.08^\circ N, 14.44^\circ E$ and the coordinates of Brno are $49.20^\circ N, 16.61^\circ E$. *Matěj was thinking about the type of projection of Google maps.*

We have to realize what Matěj's trajectory looks like. In both cases, a part of the path goes along the parallel of latitude and the other part along the meridian. The paths along the meridian are the same in both cases, because Matěj needs to cross the distance from the parallel going through Prague and the parallel going through Brno. This is not the case of the other part of the path, because the distance between two meridians is not constant. When Matěj goes east, he moves along one parallel. The radius r of the parallel depends on the latitude (we mark it as φ) as

$$r = R\cos\varphi\,,$$

where $R = 6\,378\,\mathrm{km}$ is the radius of the Earth. The distance s Matěj travelled along the parallel is given by the length of the circular arc with the angle $\Delta\lambda = \lambda_\mathrm{B} - \lambda_\mathrm{P}$, therefore

$$s = \frac{\pi}{180^\circ} r\Delta\lambda\,,$$

where we mark the latitude in degrees as λ and the factor $\frac{\pi}{180^\circ}$ converts degrees to radians. We get the difference between the paths as

$$\Delta s = s_1 - s_2 = \frac{\pi}{180^\circ}\left(r_1 - r_2\right)\Delta\lambda = \frac{\pi}{180^\circ} R\Delta\lambda\left(\cos\varphi_\mathrm{P} - \cos\varphi_\mathrm{B}\right) \doteq -2\,830\,\mathrm{m}\,.$$

We can see that even when looking on the flat map of the "small" Czech Republic there is a certain deformation, and the actual distances can differ by up to several kilometers.

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Problem X.3 ... poisoned

5 points

A certain toxic substance got into a pond with $V = 130\,\mathrm{m^3}$ of water. The only tributary of the lake is a stream with a flow rate $Q = 3.8\,\ell{\cdot}\mathrm{s^{-1}}$ and the water flowing in immediately mixes with the contaminated water in the pond. What time does it take to decrease the concentration of the substance to one tenth of the initial concentration, assuming that the water level in the pond doesn't change? *Jarda wanted to go fishing in Bečva.*

Let us denote the concentration of the toxic substance by c. If the water in the pond mixes completely with the water from the stream, the concentration of the toxic substance in the water flowing out of the pond is also c, while the water flowing into the pond has zero concentration of this substance. The constant water level means that the amount of water flowing in is the same as the amount of water flowing out. The difference in the amount of the toxic substance in the pond for a time period $\mathrm{d}t$ is

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}\left(cV\right)}{\mathrm{d}t} = V\frac{\mathrm{d}c}{\mathrm{d}t}\,.$$

During this time, water with volume $Q\,\mathrm{d}t$ will flow out, therefore the amount of the toxic substance will decrease by $\mathrm{d}n = -cQ\,\mathrm{d}t$. From this, we get a differential equation

$$V\frac{\mathrm{d}c}{\mathrm{d}t} = -cQ\,,$$

and we solve it by separation of variables

$$\frac{\mathrm{d}c}{c} = -\frac{Q}{V}\,\mathrm{d}t\,.$$

By integrating it, we get

$$c(t) = c_0\mathrm{e}^{-\frac{Q}{V}t}\,.$$

For the concentration to decrease to one tenth, $c(t) = \frac{c_0}{10}$ must apply. From this, we can express the time as

$$t = \frac{V}{Q} \ln \frac{c_0}{c} = \frac{V}{Q} \ln 10 \doteq 21.9\,\mathrm{h}\,.$$

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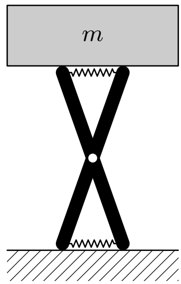
Problem X.4 ... malicious X

5 points



There are two rods in the figure, each with length $l = 1.2\,\mathrm{m}$. They are connected together in the middle in such a way that they can rotate freely. The upper ends and the lower ends of the rods are connected by two springs, each with stiffness $k = 90\,\mathrm{N{\cdot}m^{-1}}$ and rest length $s = 0.4\,\mathrm{m}$. We put a weight with mass m on the top and wait until the system reaches an equilibrium. Find a value of m such that the springs get twice as long. Assume that the system cannot fall over. *Jáchym felt bored.*

Let's denote the length of the spring after elongation by $2x$, its original length by $2x_0 = s$, the final height of the "X" by $2y$ and the length of each rod by $2r = l$. From the Pythagorean theorem, we get

$$r^2 = x^2 + y^2\,.$$

The forces, which the springs exert on each of the rods, have magnitudes $F_x = 2k\,(x - x_0)$. Gravitational (and buoyant) forces are uniformly distributed, so the force on each end of each rod is $F_y = mg/2$. The rods don't move with respect to each other when the net torque on each of them is zero. Therefore, in equilibrium,

$$xF_y = yF_x\,.$$

After we substitute for F_x and F_y, we get

$$m = \frac{4ky\,(x - x_0)}{gx}\,.$$

We are looking for such a mass m that $x = 2x_0 = s$. We can express from the Pythagorean theorem that $y = \sqrt{r^2 - 4x_0^2}$. The result is

$$m = \frac{4k\,(2x_0 - x_0)}{2gx_0}\sqrt{r^2 - 4x_0^2} = \frac{2k}{g}\sqrt{r^2 - 4x_0^2} = \frac{k}{g}\sqrt{l^2 - 4s^2} \doteq 8.21\,\mathrm{kg}\,.$$

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Problem X.5 ... slip 5 points

Jáchym found a stone in the shape of a perfect rotational ellipsoid with semi-axes $a = 5.3\,\mathrm{cm}$ and $b = 3.5\,\mathrm{cm}$, where a is the axis of rotational symetry. He let the stone slip on ice in such a way that the angle φ between the axis of rotational symmetry of the ellipsoid and the vertical axis was constant throughout the whole movement. Calculate the maximum possible magnitude of this angle φ if the mass of the stone is $m = 32\,\mathrm{g}$, the initial speed is $v = 2.79\,\mathrm{m\cdot s^{-1}}$ and the coefficient of friction between the stone and ice is $f = 0.30$. *Jáchym was behind with the third series of FYKOS, so he procrastinated by thinking about Fyziklani.*

The key to solving the problem is the stability condition for the stone considering the rotation. From the external point of view, the body is affected by three forces- the weight, reaction force from the ice field F_i and a frictional force $F_\mathrm{f} = fF_\mathrm{i}$. The sum of torques relative to the centre of the stone must be zero, which implies

$$F_\mathrm{f} r \cos\gamma = F_\mathrm{i} r \sin\gamma \,,$$

where r is the length of the line which connects the centre of the stone with the point of contact with the ice, and γ is the angle between this line and the vertical direction (see 1). We'll rewrite the equation as

$$f = \tan\gamma \,. \tag{2}$$

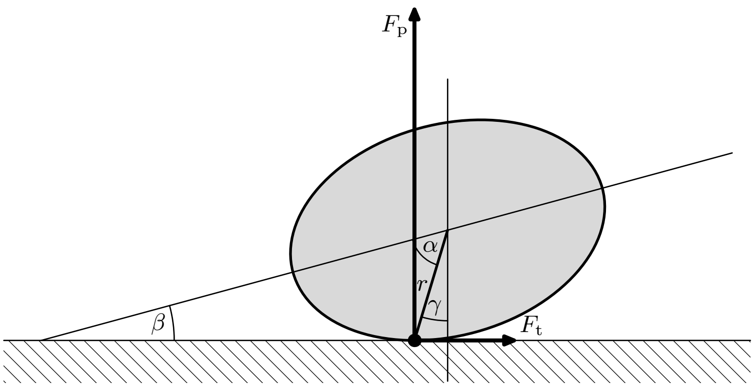

Fig. 1: Drawing of the stone in a slip.

Now let's imagine an ellipsoid rotated in the way that the semi-major axis is on the axis of x and the centre of the stone is at the origin point. The part of its surface lying in the xy plane, can be described by the function

$$y = b\sqrt{1 - \left(\frac{x}{a}\right)^2} \,,$$

as shown in image 2. We will denote the inclination of the tangent line of the function from the horizontal direction as $-\beta$. We chose the minus symbol because we know that this angle is

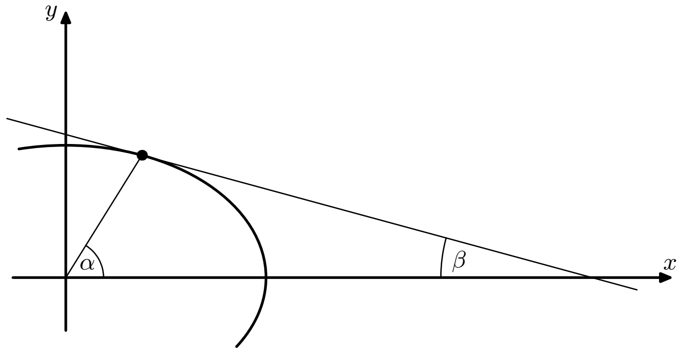

Fig. 2: Part of stone cross-section. The major axis of the ellipse is parallel to axis x, the minor one is parallel to axis y. The magnitudes of the angles α and β corresponds to the image 1.

going to be negative and we're only interested in its magnitude. The tan of this angle is equal to the function's derivative, so we have

$$y' = -b\left(1-\left(\frac{x}{a}\right)^2\right)^{-\frac{1}{2}}\frac{x}{a^2} = -\left(\frac{b}{a}\right)^2\frac{x}{y}\,.$$

For the point on the ellipsoid surface where we calculate the derivative, let's define α as the angle between the line connecting this point with the centre of the ellipsoid and the semi-major axis. Then, the following holds

$$y' = -\left(\frac{b}{a}\right)^2\frac{1}{\tan\alpha} = \tan\left(-\beta\right) = -\tan\beta\,.$$

And so we obtained the second important formula

$$\tan\beta\tan\alpha = \left(\frac{b}{a}\right)^2 = \frac{1}{k}\,, \tag{3}$$

where k isn't a constant. It is a tool of surprisal, which we will use later.

From the geometry of this problem, we realize that $\varphi = \alpha + \gamma$ a $\alpha + \beta + \gamma = \frac{\pi}{2}$. Further, we need the following trigonometric identity

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x}\,,$$
$$\tan\left(x+y\right) = \frac{\tan x + \tan y}{1 - \tan x\tan y}\,.$$

Obviously, it is true that

$$\tan\varphi = \tan\left(\alpha+\gamma\right) = \tan\left(\frac{\pi}{2} - \beta\right) = \frac{1}{\tan\beta} = k\tan\alpha\,.$$

In the last equation, we used the relation (3). The second expression we can rewrite using the formula for the tangent of a sum, and we get

$$\begin{aligned} \tan\alpha + \tan\gamma &= k\tan\alpha\left(1 - \tan\alpha\tan\gamma\right) , \\ \tan\alpha + f &= k\tan\alpha\left(1 - f\tan\alpha\right) , \\ 0 &= fk\tan^2\alpha + (1-k)\tan\alpha + f , \end{aligned}$$

where we have just substituted $\tan\gamma$ by (2). The solution of this quadratic equation is

$$\tan\alpha = \frac{k - 1 \pm \sqrt{(k-1)^2 - 4f^2k}}{2fk} ,$$

from where we get the result thanks to the formula $\tan\varphi = k\tan\alpha$. We can write

$$\begin{aligned} \varphi &= \arctan\left(\frac{1}{2f}\left(k - 1 \pm \sqrt{(k-1)^2 - 4f^2k}\right)\right) = \\ &= \arctan\left(\frac{1}{2f}\left(\left(\frac{a}{b}\right)^2 - 1 \pm \sqrt{\left(\left(\frac{a}{b}\right)^2 - 1\right)^2 - 4f^2\left(\frac{a}{b}\right)^2}\right)\right) . \end{aligned}$$

Because we are looking for the biggest possible angle, we choose the root with the plus sign. After plugging in the numbers, we get $\varphi \doteq 74.8^\circ$. We can see that this solution corresponds to a stable equilibrium position. Choosing the root with the minus sign would result in an unstable equilibrium.

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Problem X.6 ... cruel game of fate 5 points

Mr. A and Mr. B were born on the same day in a dystopian future inspired by Orwell's novel 1984. Mr. A is an ordinary citizen, while Mr. B is a member of the Inner Party.

The thought police is making up ficticious charges. Mr. A noticed that the "half-life" of an ordinary citizen is 5 years and he will surely meet the same fate.

The Big Brother organizes a purge in the Party once every 5 years. During that, 60 % of the party members are randomly selected and removed. That may be the fate of Mr. B.

What is the probability that Mr. A will live longer than Mr. B if one purge has just ended? Assume that purges and the thought police are the only causes of death.

Jindra can finally say that mandatory reading in school was useful.

Let's denote the 5-year time interval by T. Next, let $a = 0.5$ and $b = 0.4$ be the fractions of survivors for the two ways to die, respectively.

The probability that Mr. B dies during the n-th purge ($n = 1$ is the first purge from today, in five years) is

$$P_\mathrm{B}(n) = b^{n-1} - b^n .$$

The probability that Mr. A dies after the n-th purge is

$$P_\mathrm{A}(n) = a^{\frac{nT}{T}} = a^n ,$$

so the compound probability that Mr. B dies during the n-th purge and Mr. A dies later is the product of these two probabilities (the two events are independent)

$$P(n) = \left(b^{n-1} - b^n\right) a^n = \frac{1-b}{b} (ab)^n \ .$$

If we want to know the total probability that Mr. A lives longer than Mr. B, we have to calculate a sum over all n

$$P(T_\mathrm{A} > T_\mathrm{B}) = \frac{1-b}{b} \sum_{n=1}^{\infty} (ab)^n = \frac{1-b}{b} \left(\frac{1}{1-ab} - 1\right) = \frac{a\,(1-b)}{1-ab} = \frac{3}{8} \, .$$

The probability that Mr. A lives longer than Mr. B is 3/8.

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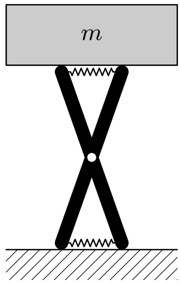
Problem X.7 ... even more malicious X 5 points



There are two rods in the figure, each with length $l = 1.2\,\mathrm{m}$. They are connected together in the middle in such a way that they can rotate freely. The upper ends and the lower ends of the rods are connected by two springs, each with stiffness $k = 90\,\mathrm{N{\cdot}m^{-1}}$ and rest length $s = 0.4\,\mathrm{m}$. We put a weight with mass $m = 7\,\mathrm{kg}$ on the top of this system. What is the difference between the highest and the lowest possible stable equilibrium position of the weight? Do not consider the position where the weight is on the ground. The coefficient of friction between the rods and the weight, and also between the rods and the ground, is $f = 0.6$. Assume that the system cannot fall over, its mass is negligible compared to the mass of the weight and the weight is longer than the rods. *Jáchym felt even more bored.*

In the equilibrium position, the total force moments applied to each of the rods is equal to zero. The range of possible equilibrium positions will be caused by the fact that the frictional force doesn't have any specific magnitude, but it is just big enough for the body not to move.

Mark the spring length after extension $2x$, the initial length $2x_0 = s$, the construction height $2y$ and the rod length $2r = l$. The elastic force acting at the end of the rod will be $F_x = 2k(x - x_0)$, while the weight and pressure force of the pad will be $F_y = mg/2$. The magnitude of force of friction will lie in the interval $\langle -F_\mathrm{t}, F_\mathrm{t} \rangle$, where $F_\mathrm{t} = fF_y$. For the total torque in the maximum and minimum position we can apply the formula

$$xF_y - y\left(F_x + zF_\mathrm{t}\right) = 0 \, ,$$

where $z \in \langle -1, 1 \rangle$. By substituting in the forces

$$xmg - y\left(4k\left(x - x_0\right) + zfmg\right) = 0 \, ,$$

Trivially, we can write $r^2 = x^2 + y^2$, from which we can express x as a function of y. However, it is immediately apparent that the resulting relation will be a high degree polynomial and so trying to solve it analytically would be impractical. Therefore, we will take a different approach – we will express z as a function of y and we will be interested in what points the condition $z \in \langle -1, 1 \rangle$ holds. We know that

$$z = \frac{1}{f}\left(\sqrt{\frac{r^2}{y^2} - 1} - \frac{4k}{mg}\left(\sqrt{r^2 - y^2} - x_0\right)\right) = \frac{1}{f}\left(\sqrt{\frac{l^2}{h^2} - 1} - \frac{2k}{mg}\left(\sqrt{l^2 - h^2} - s\right)\right),$$

where we have defined the required height as $h = 2y$. The function $z(h)$ is shown in graph 3. The minimum and maximum heights are

$$h_1 \doteq 0.451\,\mathrm{m}\,,$$
$$h_2 \doteq 1.173\,\mathrm{m}\,.$$

The answer is therefore $\Delta h = h_2 - h_1 \doteq 0.722\,\mathrm{m}$.

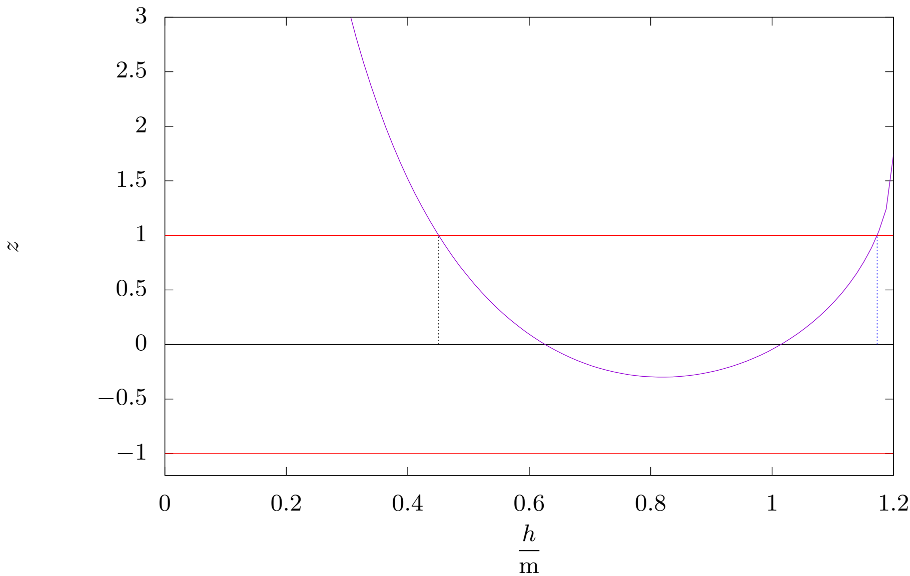


Fig. 3: Function $z(h)$. Horizontal lines indicate the interval $\langle -1, 1 \rangle$.

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Problem A.1 ... flight from the Sun 5 points

At what speed does an airplane need to fly (with respect to the surface of the Earth) at a height $h = 10\,\mathrm{km}$ *straight along a 42*$^{\mathrm{nd}}$ *parallel of latitude if it wants to keep flying away from sunrise (i.e. to always be just above the place where the Sun rises)?*

Danka was travelling by plane with the Sun rising behind her.

If we think about the situation carefully, we realise that the plane must always be in the same point relative the Earth and the Sun. Therefore the plane moves relative to the surface of the Earth the same way as the surface of the Earth moves relative to the centre of the Earth and to the Sun. We can find such velocity as a ratio between the circumference o of the Earth at the latitude given (42^{nd} parallel) and the period of the Earth rotation $T = 24\,\mathrm{h}$ relative to the Sun. The length of the parallel can be calculated easily as $o = 2\pi r$, where r is the radius of the parallel. We obtain it from the radius of the Earth $R_{\mathrm{E}} = 6\,378\,\mathrm{km}$ as

$$r = R_{\mathrm{E}} \cos\alpha \,,$$

where α is the angle between the equator and the parallel line, which is $42\,^{\circ}$. We obtain, that the velocity of the rotation of the Earth on the 42^{nd} parallel line towards the plane is

$$v = \frac{o}{T} = \frac{2\pi R_{\mathrm{E}} \cos\alpha}{T} \,.$$

Plugging in the numerical values, we get

$$v \doteq 1\,240\,\mathrm{km{\cdot}h^{-1}} \,.$$

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Problem A.2 ... SETI frequency 5 points

SETI is a project trying to find evidence of extraterrestrial civilisations by observing cosmic radio signals. In 1973, Drake and Sagan proposed searching on some "natural frequency" obtained by a combination of the Boltzmann constant k_{B}*, Planck constant* h *and the current temperature of cosmic microwave background* $T_0 = 2.725\,\mathrm{K}$*. Find a formula for a frequency using these three constants, supposing that the dimensionless constant is 1. Calculate this frequency.*

Karel was reading a book about the SETI project.

The solution to this problem is found by dimensional analysis. We can easily guess the formula for the frequency – it is not that hard – but we are going to show a proper solution using a system of linear equations. We are trying to find parameters α, β, γ such that

$$f = k_{\mathrm{B}}^{\alpha} \cdot h^{\beta} \cdot T_0^{\gamma} \,.$$

The unit on the left-hand side must be the same as the unit on the right-hand side. We express the three constants using basic SI units kg, m, s and K

$$\mathrm{Hz} = \left(\mathrm{J{\cdot}K^{-1}}\right)^{\alpha} \cdot \left(\mathrm{J{\cdot}s}\right)^{\beta} \cdot \left(\mathrm{K}\right)^{\gamma} \,,$$
$$\mathrm{s^{-1}} = \left(\mathrm{kg{\cdot}m^{2}{\cdot}s^{-2}{\cdot}K^{-1}}\right)^{\alpha} \cdot \left(\mathrm{kg{\cdot}m^{2}{\cdot}s^{-1}}\right)^{\beta} \cdot \left(\mathrm{K}\right)^{\gamma} \,,$$

$$\begin{aligned} 0 &= \alpha + \beta\,, \\ 0 &= 2\alpha + 2\beta\,, \\ -1 &= -2\alpha - \beta\,, \\ 0 &= -\alpha + \gamma\,. \end{aligned}$$

We got a system of four linear equations with three unknowns. However, two of the equations are linearly dependent, so we have effectively three equations with three unknowns. The solution of this system of linear equations is $\alpha = 1$, $\beta = -1$, $\gamma = 1$ (you can check that it is correct). The frequency is

$$f = \frac{k_\mathrm{B} T_0}{h} \doteq 56.78\,\mathrm{GHz}\,.$$

The ”natural frequency” is 56.78 GHz, which corresponds to the wavelength 5.3 mm in vacuum.

It is very interesting that this frequency is universal if we assume that the temperature of the universe was homogeneous enough during the recombination epoch and that the expansion of the universe is homogeneous and isotropic. Taking into account the accuracy of our calculation, these assumptions are accurate. Due to the expansion of the universe, wavelengths grow longer with time, but the temperature of cosmic microwave background is decreasing at the same rate. If we sent a signal on this ”natural frequency”, its wavelength would increase with time; however, possible observers would observe on the ”natural frequency” based on their measured temperature of cosmic microwave background, so they could identify our signal.

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Problem A.3 ... gravitation in The Orwille — 5 points

Isaac, a character in the TV series The Orwille, says that the planet Kaylon 1 has a circumference $o = 57\,583\,\mathrm{km}$, an average density $\rho = 4.42\,\mathrm{g{\cdot}cm^{-3}}$ and ”gravity” 1.13 times higher than that of Earth. Assume that by ”gravity”, he means the gravitational acceleration on the planet's surface and use the value $a_\mathrm{g} = 11.11\,\mathrm{m{\cdot}s^{-2}}$. What is the corresponding value of the gravitational constant, assuming that the planet is a homogenous sphere and the laws of physics work the same way as in reality? *Karel was watching The Orwille.*

The gravitational acceleration at the planet's surface is

$$a_\mathrm{g} = G\frac{M}{R^2}\,, \tag{4}$$

where G is the gravitational constant that we want to find, M is the mass of the planet and R is its diameter. The mass of a homogeneous sphere is simply

$$M = \rho V = \frac{4}{3}\rho\pi R^3\,,$$

where V is its volume. Next, we want to express G as a function of the circumference of the Earth instead of its radius, so we use $o = 2\pi R$. Plugging into (4), we get

$$\begin{aligned} G &= a_\mathrm{g}\frac{R^2}{M} = a_\mathrm{g}\frac{R^2}{\frac{4}{3}\rho\pi R^3} = a_\mathrm{g}\frac{3}{4\pi\rho R} = \\ &= \frac{3}{2}\frac{a_\mathrm{g}}{\rho o} \doteq 6.55\cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}\,. \end{aligned}$$

The gravitational constant in such a universe would be $6.55 \cdot 10^{-11}\,\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$. Gravitation is then approximately 2 % weaker than in our universe.

It is interesting that the number π did not show up in the result, so even if it was different in the other universe, but the same laws of physics applied, we would not recognise the difference.

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Problem A.4 ... a little crescent moon

5 points

What is the solid angle in the sky occupied by the illuminated part of the Moon if it is visible at an angular distance (elongation) $\alpha = 30^\circ$ from the Sun? The angular diameter of the Moon in the sky is $\theta = 0.52^\circ$. *Matěj was looking dreamily at the Moon.*

Assume that the Moon is far from the Earth and the Sun is far enough from the Moon. Since the angular diameter of the Moon is small, we can easily calculate the solid angle Ω_0 that it occupies in the sky (including its non-illuminated part) just as the surface of a circle of the diameter θ

$$\Omega_0 = \frac{\pi\theta^2}{4}\,.$$

Now we need to find the relative size of the illuminated part of the visible side of the Moon. From the Earth, we observe the circle under an angle and its projection into the plane perpendicular to the observed direction is an ellipse. In the picture we see, that the semi-minor axis of the ellipse is $b = r\sin(\beta - \pi/2) = -r\cos\beta$, where r is the radius of the Moon. The relative share k of the illuminated part of the Moon can be easily expressed as the ratio of the illuminated part (see the picture) and the total projection of the Moon into the observed direction

$$k = \frac{\frac{\pi r^2 - \pi r b}{2}}{\pi r^2} = \frac{r-b}{2r} = \frac{1+\cos\beta}{2}\,.$$

The angle β is the angle Earth-Moon-Sun. It could be easily calculate using the law of cosines, but since the Sun is very far from both the Earth and the Moon, we can approximate

$$\beta \approx \pi - \alpha\,.$$

Finally, the solid angle of the visible part is

$$\Omega = k\Omega_0 = \frac{1-\cos\alpha}{2}\frac{\pi\theta^2}{4} = \frac{\pi\theta^2}{8}(1-\cos\alpha) = 0.0142\,\mathrm{deg}^2 = 4.33 \cdot 10^{-6}\,\mathrm{sr}\,.$$

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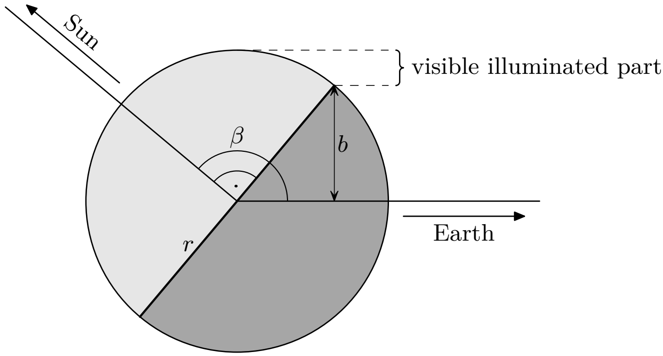

Problem A.5 ... a New Year's comet 5 points

Recently, specifically on December 20th 2020, we could see a particular comet in the sky for the first time in 14 years. It appeared at exactly the same spot as on December 20th 2006, the year when the first Fyziklani ever took place. Under what angle does the comet's orbit intersect the orbit of Earth if its eccentricity is 0.9? You may neglect the gravitational influence of Earth on the comet and assume the orbit of Earth to be perfectly circular. The comet's orbit is in the same plane as that of Earth. *Jarda is fascinated by comets.*

According to the assignment, the situation repeats exactly after 14 years, which is also the comet's orbital period. From Kepler's third law, the semi-major axis of the comet's orbit is equal to $a = T^{\frac{2}{3}}$, where the period T is in years and the semi-major axis of the ellipse a in astronomical units, giving us $a \doteq 5.809\,\mathrm{AU}$. The angle is determined by the law of conservation of angular momentum, which holds in a central gravitational field.

When moving on an ellipse, the law of conservation of energy has the form

$$E = -\frac{MmG}{2a}\,,$$

where M is the mass of the central body (in this problem it's the Sun), m is the mass of the comet, G is the gravitational constant and a is the semi-major axis length. The Sun is in one of the focal points of the ellipse. From the ellipse geometry, we are able to find the distance in perihelion (i.e. when the comet is closest to the Sun). This is

$$r_\mathrm{p} = a\left(1 - e\right)\,,$$

where $e = 0.90$ is the eccentricity of the comet's orbit. Using the law of conservation of energy, we can now find the kinetic energy of the comet as

$$E_\mathrm{k} = E - E_\mathrm{p} = -\frac{MmG}{2a} - \left(-\frac{MmG}{r_\mathrm{p}}\right) = (MmG)\left(\frac{1}{r_\mathrm{p}} - \frac{1}{2a}\right)\,.$$

We can express the velocity as

$$v_\mathrm{p} = \sqrt{2MG}\sqrt{\frac{1}{r_\mathrm{p}} - \frac{1}{2a}} = \sqrt{2MG}\sqrt{\frac{1+e}{2a\left(1-e\right)}}\,,$$

where we plugged in the expression for r_p. Angular momentum has magnitude

$$L = mvr\sin\alpha\,,$$

where α is the angle that the line joining the orbiting body and the Sun makes with the velocity vector (which is always a tangent to the trajectory). If the comet is in perihelion, then the velocity vector is perpendicular to this line and the angular momentum is

$$L = mv_\mathrm{p}r_\mathrm{p} = m\sqrt{MGa\left(1-e^2\right)}\,.$$

In a similar way, we calculate the speed of the comet at the moment of intersection with Earth's orbit, at the distance $r = 1\,\mathrm{AU}$ from the Sun. It is

$$v_\mathrm{E} = \sqrt{2MG}\sqrt{\frac{1}{r}-\frac{1}{2a}} = \sqrt{MG}\sqrt{\frac{2a-r}{ar}}\,.$$

From the law of conservation of angular momentum we can find the angle α as

$$\sin\alpha = \frac{v_\mathrm{p}r_\mathrm{p}}{v_\mathrm{E}r_\mathrm{z}} = \frac{\sqrt{MGa\left(1-e^2\right)}}{r\sqrt{MG}\sqrt{\frac{2a-r}{ar}}} = \frac{a\sqrt{\left(1-e^2\right)}}{\sqrt{r\left(2a-r\right)}}\,,$$
$$\alpha \doteq 51.0^\circ.$$

This angle is, therefore, made by the speed and the line connecting the comet and the Sun. But our goal was to find the angle between the velocity vector and the Earth's trajectory. It is

$$\beta = 90^\circ - \alpha \doteq 39.0^\circ\,.$$

The comet's trajectory intersects the Earth's orbit at the angle 39.0°.

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Problem A.6 ... deadly Sun — 5 points

An astronomer noticed a sudden brightening on the surface of the Sun near a sunspot. This coronal mass ejection released a proton with total energy $E = 1\,012\,\mathrm{MeV}$. How long does it take for the proton to fly to Earth (from the time when the astronomer noticed the eruption)?

Dodo wanted to observe, but it was misty.

The required time is given by the difference of flight times of the particle and a photon

$$\Delta T = \frac{l}{v} - \frac{l}{c}\,,$$

where $l = 149.6\cdot 10^9\,\mathrm{m}$ is the distance between the Earth and the Sun. The magnetic fields will have little effect on a proton with such high energy, and so we can assume that it was flying directly to Earth. Further, we can verify that the loss of energy due to gravity is negligible. The velocity of the proton can be determined from the relativistic relation for energy.

$$E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}\,,$$

where the rest energy m_0c^2 is the rest mass-energy of proton $m_\mathrm{p} = 938.27\,\mathrm{MeV}$. By expressing the velocity and substituting it into the relationship for the time difference we get

$$\Delta T = \frac{l}{c}\left(\frac{c}{v} - 1\right) = \frac{l}{c}\left[\left(1 - \frac{m_\mathrm{p}^2}{E^2}\right)^{-\frac{1}{2}} - 1\right] \doteq 833\,\mathrm{s}\,.$$

From the moment the flash is observed, the particle will reach the Earth in about a quarter of an hour.

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Problem C.1 ... non-ideal ammeter

5 points

We have a resistor with resistance $R = 100\,\Omega$. We connect the resistor in series with an ammeter and connect them both in parallel with an ideal voltmeter. Then, we power the circuit by a DC voltage source. The ammeter shows $I = 12\,\mathrm{mA}$ and the voltmeter shows $U = 1.3\,\mathrm{V}$. What is the resistance of the ammeter? *Legolas was measuring resistances as part of lab work.*

The ammeter is connected in series with the resistor, so the value $I = 12\,\mathrm{mA}$ is the true current in the resistor. The voltage on the resistor is therefore $U_\mathrm{r} = IR$.

However, the voltmeter is connected in parallel to both the resistor and the ammeter, so the voltage U measured by the voltmeter is the sum of the voltage on the resistor U_r and the voltage on the ammeter U_a. Remember that the voltmeter is ideal, so no current flows through it. The voltage on the ammeter is $U_\mathrm{a} = U - U_\mathrm{r}$.

The current in the ammeter is the same as the current in the resistor, so the resistance of the ammeter is

$$R_\mathrm{a} = \frac{U_\mathrm{a}}{I} = \frac{U - U_\mathrm{r}}{I} = \frac{U - IR}{I} = \frac{U}{I} - R \doteq 8.3\,\Omega\,.$$

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Problem C.2 ... cuboids

5 points

In his old box, Kiko discovered five identical conductive blocks (cuboids) with edge lengths $a = 5.0\,\mathrm{cm}$, $b = 4.0\,\mathrm{cm}$ and $c = 3.0\,\mathrm{cm}$. Each of them had resistivity $\rho = 1.21 \cdot 10^{-6}\,\Omega{\cdot}\mathrm{m}$. Kiko connected the blocks a few times and each time, he measured the total resistance of the circuit he made. What is the lowest resistance he could have measured, assuming that he always used all of the blocks? He always connected the blocks to the circuit using two parallel conductive panels that weren't touching each other. *Kiko was playing with bricks.*

We obtain the smallest resistance of a block when the current flows through the widest possible cross-section along the shortest possible path. Comparing the individual combinations of possible rotation of the block we find out that the smallest resistance is

$$R_\mathrm{min} = \rho \frac{c}{ab}\,.$$

The solution for the ideal connection of smallest resistance is now simple. We connect them all in parallel, which gives the equation for the total resistance of the circuit

$$\frac{1}{R_c} = \frac{5}{R_\mathrm{min}}\,.$$

By inverting the equation we get the result

$$R_c = \frac{R_\mathrm{min}}{5} = 3.63 \cdot 10^{-6}\,\Omega\,.$$

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Problem C.3 ... inductors 5 points

Káťa wanted to make a coil with an inductance that would be as high as possible, but she was also attempting to keep its ohmic resistance low. The old inductor had a radius $r_0 = 2\,\mathrm{cm}$ and $N_0 = 100$ turns which were touching each other but didn't overlap. Káťa used the wire from this coil to make a new coil with a radius $r_1 = 4\,\mathrm{cm}$. The turns on the new coil were also right next to each other and they didn't overlap. By how many percent did the inductance rise? *Káťa was pursuing an internship at a tokamak but she ended up winding coils.*

The inductance L of a long solenoid is

$$L = \frac{\mu N^2 S}{l}\,,$$

where l is length of the solenoid, S is its cross-sectional area and μ is permeability of the environment. This coil doesn't possess a core, so we can assume that μ is equal to the permeability of vacuum. The ratio of inductances is

$$\frac{L_1}{L_0} = \frac{\mu N_1^2 \pi r_1^2 l_0}{\mu N_0^2 \pi r_0^2 l_1} = \left(\frac{N_1 r_1}{N_0 r_0}\right)^2 \frac{l_0}{l_1} = 2\,,$$

where we noticed that $N_1 r_1 = N_0 r_0$ is proportional to the total length of the wire and the length of the coil l is proportional to the number of turns. The new coil, whose radius is twice as big, has half as many turns compared to the old one. The inductance of the new coil is therefore twice as big – in other words, it rises by $100\,\%$. It seems like we could increase the inductance arbitrarily by using a large radius. However, for large S/l, the relation $L = \mu N^2 S/l$ is not valid anymore.

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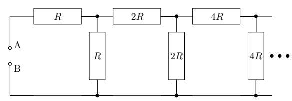
Problem C.4 ... infinite and nasty circuit 5 points



In the figure, we can see part of an infinite circuit of resistors. The resistances of resistors in it are terms of a geometric series. In other words, in the first block, each resistance is R, in the second block, it is $2R$ and in each following block, the resistances are twice as large as for resistors in the previous block. What is the total resistance between points A and B? The answer should be a multiple of R.

Karel was modifying problems on infinite circuits.

In problems with an infinite circuit, the trick is to find the repeating pattern and then to form a suitable equation from which we can (hopefully easily) express the desired total resistance R_∞. In our case, we can notice that by removing the first two resistors, we get a very similar circuit. The only difference is that each resistor in that circuit has twice as large resistance as before. That means the whole circuit has the resistance $2R_\infty$. We can draw the circuit in two equivalent ways, as shown in the figure 4. We get the equation

$$R_\infty = R + \frac{2RR_\infty}{R + 2R_\infty}\,.$$

We can easily convert it to a quadratic equation, which we solve

$$RR_\infty + 2R_\infty^2 = R^2 + 2RR_\infty + 2RR_\infty\,,$$
$$0 = 2R_\infty^2 - 3RR_\infty - R^2\,,$$
$$R_\infty^{\pm} = \frac{3R \pm \sqrt{9R^2 + 8R^2}}{4} = \frac{3 \pm \sqrt{17}}{4}R\,.$$

The physically correct solution is the one with the positive sign. The other one would result in negative resistance. The only solution is the total resistance

$$R_\infty = \frac{3 + \sqrt{17}}{4} R \doteq 1.78\,R\,.$$

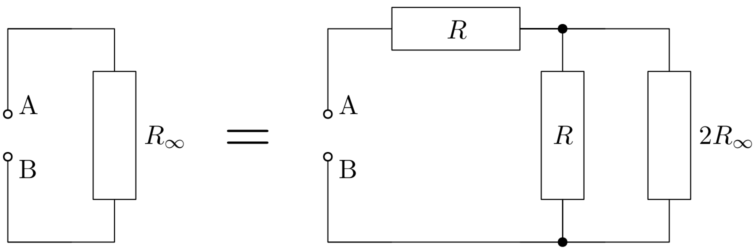


Fig. 4: Two equivalent circuits.

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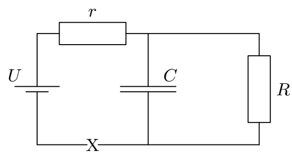
Problem C.5 ... long wiring 5 points



Suppose that we have an electric appliance plugged to a $U_0 = 120\,\mathrm{V}$ *DC power source via a long cable. Since the conductors are close to each other, we create an electric scheme of the circuit shown in the figure. The resistor* R *is the appliance, the resistor* $r \ll R$ *represents the resistance of the cable and the capacitor* $C = 420\,\mathrm{nF}$ *represents its capacitance. The point denoted by "X" suffers from improper contact, which leads to periodically switching the circuit on and off. Each of these states lasts for the same constant time* $T = 0.001\,\mathrm{s}$*. Find the power that the power source will supply to the appliance over a very long time after plugging the circuit in. If the connection was stable, the power supplied by the source would be* $P_0 = 30\,\mathrm{W}$*.*

Dodo was fighting with an extension lead.

First, we will solve for the current in the circuit. From Kirchhoff laws we know that current I through the battery and resistor r is equal the sum of currents I_C and I_R through the capacitor and the resistor R. The voltage on the appliance and the capacitor is the same, therefore we have

$$RI_R = U_C\,.$$

Taking the derivative with respect to time using $C\,\mathrm{d}U = \mathrm{d}Q$ we get

$$RC\dot{I}_R = I_C\,,$$

where the dot over I_R means time derivative. Final basic relation we are going to use is for voltage distribution over the resistors

$$U_0 = rI + RI_R\,.$$

Whenever the circuit is interrupted at point X, the capacitor is being discharged through the appliance. For battery curent $I = 0$, therefore $I_R = -I_C$ and

$$-RC\dot{I}_C = I_C\,.$$

The solution is the decresing exponencial relation

$$I_C^{\mathrm{v}}(t) = I_C^{\mathrm{v}}(0)\,\mathrm{e}^{-\frac{t}{RC}}\,.$$

When the battery is connected we have

$$RI_R + rI_C + rI_R = U_0\,,$$

from which, by differentiating, we obtain

$$\frac{R+r}{RC}I_C + r\dot{I}_C = 0\,,$$

with has a solution

$$I_C^{\mathrm{z}} = I_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\,.$$

The next step is to make sure these solutions match in the instants of disconnecting and reconnecting the battery. The voltage on the capacitor must always change continuously (otherwise the there would be an infinite flow of energy). This voltage is “measured” on the appliance, therefore the current I_R is continuous. At the moment of disconnecting the battery

$$I_R^{\mathrm{z}}(T) = I_R^{\mathrm{v}}(0)\,,$$

and when reconnecting

$$I_R^{\mathrm{v}}(T) = I_R^{\mathrm{z}}(0)\,.$$

We can determine the current through the resistor from the expressions above

$$I_R^{\mathrm{v}} = -I_C^{\mathrm{v}}\,,$$
$$I_R^{\mathrm{z}} = \frac{U_0 - rI_C^{\mathrm{z}}}{R+r}\,.$$

Substituting the solutions into the preceeding we obtain a system of equations for $I_C^{\mathrm{v}}(0)$ and $I_C^{\mathrm{z}}(0)$

$$\frac{U_0}{r} + \frac{R+r}{r}I_C^{\mathrm{v}}(0) = I_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\,,$$
$$I_C^{\mathrm{z}}(0) = \frac{U_0}{r} + \frac{R+r}{r}I_C^{\mathrm{v}}(0)\,\mathrm{e}^{-\frac{T}{RC}}\,.$$

By addition of these equation we get the conservation equation for charge

$$\frac{R+r}{r} I_C^{\mathrm{v}}(0)\left(1-\mathrm{e}^{-\frac{T}{RC}}\right) = -I_C^{\mathrm{z}}(0)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right) .$$

Substituting the second equation into the first

$$I_C^{\mathrm{v}}(0) = -\frac{U_0}{R+r}\frac{1-\mathrm{e}^{-\frac{R+r}{r}\frac{T}{RC}}}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}} ,$$

and using the preceeding formula

$$I_C^{\mathrm{z}}(0) = \frac{U_0}{r}\frac{1-\mathrm{e}^{-\frac{T}{RC}}}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}} .$$

We can see that discharging current has negative sign, consistent with our choice of direction of current.

This is the full solution of the circuit, we just need to find the power. Work done by the battery per one cycle is

$$W = \int_0^T U_0 I^{\mathrm{z}}(t)\,\mathrm{d}t + \int_0^T U_0 I^{\mathrm{v}}(t)\,\mathrm{d}t = \int_0^T U_0 I^{\mathrm{z}}(t)\,\mathrm{d}t .$$

Battery current is

$$I^{\mathrm{z}} = I_C^{\mathrm{z}} + I_R^{\mathrm{z}} = I_C^{\mathrm{z}} + \frac{U_0 - rI_C^{\mathrm{z}}}{R+r} = \frac{U_0 + RI_C^{\mathrm{z}}}{R+r} ,$$

which gives us the work as

$$\begin{aligned} W &= \frac{U_0}{R+r}\int_0^T \left(U_0 + RI_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\right)\mathrm{d}t = \\ &= \frac{U_0}{R+r}\left[U_0 t - \left(\left(\frac{R}{r}+1\right)\frac{1}{RC}\right)^{-1} RI_C^{\mathrm{z}}(0)\,\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{t}{RC}}\right]_0^T = \\ &= \frac{U_0}{R+r}\left(U_0 T + \frac{rR^2C}{R+r} I_C^{\mathrm{z}}(0)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)\right) = \\ &= \frac{U_0^2}{R+r}\left(T + \frac{R^2C}{R+r}\frac{\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}}\right) . \end{aligned}$$

For average power of the circuit using $P_0 = \frac{U_0^2}{R+r}$ we get

$$P = \frac{W}{2T} = \frac{P_0}{2}\left(1 + \frac{R^2C}{T(R+r)}\frac{\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\left(1-\mathrm{e}^{-\left(\frac{R}{r}+1\right)\frac{T}{RC}}\right)}{1-\mathrm{e}^{-\left(\frac{R}{r}+2\right)\frac{T}{RC}}}\right) .$$

Furthermore, for $R \gg r$ we get a nicer expression

$$P \approx \frac{P_0}{2}\left(1 + \frac{RC}{T}\left(1-\mathrm{e}^{-\frac{T}{RC}}\right)\right) ,$$

where we can see, that it lies in between $P_0/2$ for small RC/T and P_0 for RC/T going to infinity. This makes sense a for a small capacity, the circuit is the same as without the capacitor, and conversely, for high capacity the circuit is always on. This is a common use of capacitors – to overcome short outages in driving voltage. Knowing the power and voltage for the circuit when the battery is always connected, we have the resistance of the appliance

$$R \approx \frac{U_0^2}{P_0}\,,$$

and therefore the power in question

$$P = \frac{P_0}{2} + \frac{U_0^2 C}{2T}\left(1 - \mathrm{e}^{-\frac{P_0 T}{U_0^2 C}}\right) \doteq 18.0\,\mathrm{W}\,.$$

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Problem O.1 ... clapping at a concert 5 points

People are watching a concert of a rockstar. The length of the arena, i.e. distance from the closest row of attendees to the most distant row, is $l = 200\,\mathrm{m}$. The singer waves his arm whenever he wants the audience to clap once. Find the highest possible number of arm-waves per minute such that the resulting series of claps do not overlap in the singer's ears.

Dodo was listening to echo at Výstaviště Praha.

All audience members clap at the same time when they see the singer wave his arm (we can neglect the difference caused by a finite speed of light). However, we can't neglect the effect of a finite speed of sound. The time difference t between the singer hearing claps from the first row and claps from the last row is

$$t = \frac{l}{c}\,,$$

where we use the speed of sound $c = 343\,\mathrm{m{\cdot}s^{-1}}$. If the singer wants to distinguish the series of claps, the highest frequency with which he can wave his arm is

$$f = \frac{1}{t} = \frac{c}{l} \doteq 103\,\mathrm{min^{-1}}\,.$$

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Problem O.2 ... a bubble in ice 5 points

Jindra was examining an ice-covered pond. He was intrigued by an air bubble in the ice. In reality, the bubble is $a = 4.0\,\mathrm{cm}$ deep in the ice. However, different refractive indices distort the observed distance. The refractive index of ice is $n_\mathrm{i} = 1.31$, the refractive index of air is $n_\mathrm{a} = 1.00$. What is the apparent depth of the bubble that Jindra observes if he is staring at the bubble straight from above, in the direction perpendicular to the surface of the ice?

Some people ice-skate, Jindra instead observes bubbles.

Rays coming from the bubble are refracted at the ice-air boundary. The refractive index of air is lower than that of ice, so the angle of refraction is higher than the angle of incidence. Therefore, the apparent depth is smaller than the real depth. Because Jindra is looking along the normal to the surface of the ice, we will use the small angle approximation.[2] Let's pick one of the rays and denote the distance between Jindra's line of sight and the point where the ray intersects the ice-air boundary by h. The angle of incidence (inside the ice) is approximately $\varphi_\mathrm{l} = h/a$, whereas the angle of refraction is $\varphi_\mathrm{v} = h/a'$, where a' is the sought apparent depth. The angles φ_l and φ_v are related by Snell's law

$$n_\mathrm{i}\varphi_\mathrm{l} = n_\mathrm{a}\varphi_\mathrm{v}\,.$$

We can express

$$\varphi_\mathrm{v} = \frac{n_\mathrm{i}}{n_\mathrm{a}}\varphi_\mathrm{l} = \frac{n_\mathrm{i}}{n_\mathrm{a}}\frac{h}{a}\,,$$

$$a' = \frac{h}{\varphi_\mathrm{v}} = \frac{n_\mathrm{a}}{n_\mathrm{i}}a \doteq 3.05\,\mathrm{cm}\,.$$

[2] In the limit $\varphi \to 0$, we can write $\sin\varphi = \tan\varphi = \varphi$.

Jindra observes the bubble at the apparent depth $a' \doteq 3.05\,\mathrm{cm}$.

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Problem O.3 ... at a concert 5 points

Danka and Dano were standing at a distance $r_1 = 60\,\mathrm{m}$ from a loudspeaker at a rock concert. They heard the music with a sound pressure level $L_1 = 80\,\mathrm{dB}$. What is the difference (in dB) between the sound pressure level heard by them and by the people standing in front of the stage at a distance $r_2 = 10\,\mathrm{m}$ from the loudspeaker? *Danka and Dano went to a concert.*

The physical quantity which describes the power of sound is the sound pressure level L. This quantity is defined as

$$L = 20\log\frac{p}{p_0}$$

and its unit is decibel. Let p be pressure of the acoustic wave at a given point and $p_0 = 2\cdot 10^{-5}\,\mathrm{Pa}$ be the threshold of hearing. The pressure is proportional to the applied force, which is proportional to the displacement and therefore also to amplitude, which is proportional to the square root of the energy flux density of the waves. Assuming constant power of the speaker, the energy flow through any sphere around it is constant, independent of the radius r. Therefore, the energy flux density is inversely proportional to the surface of the sphere. Mathematically written,

$$p \propto F \propto I \propto \sqrt{\rho_E} \propto \sqrt{S^{-1}} \propto r^{-1}\,.$$

The ratio of the pressures at two different distances satisfies

$$\frac{p_2}{p_1} = \frac{r_1}{r_2}\,.$$

The difference in loudness we're looking for is

$$L_2 - L_1 = 20\left(\log\frac{p_2}{p_0} - \log\frac{p_1}{p_0}\right) = 20\log\frac{p_2}{p_1} = 20\log\frac{r_1}{r_2} \doteq 16\,\mathrm{dB}\,.$$

People standing directly in front of the stage hear the concert approximately 16 dB louder.

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Problem O.4 ... bells 5 points

A church stands at a distance $h = 100\,\mathrm{m}$ from an infinite wall. The church bell rings with a period $T_0 = 1.00\,\mathrm{s}$. Kiko is walking around the church at a constant distance l from it. As he walks, he realises that the time interval between each bell ring and its echo is changing. What is the minimum distance l such that he could hear ringing (without distinguishing between the original sound and the echo) with a period $T = 0.50\,\mathrm{s}$ at some point of his walk. The speed of sound is $v = 310\,\mathrm{m\cdot s^{-1}}$. Neglect the height of the church tower with the bell.

Kiko was visiting a midnight mass.

We can substitute the echo by a church which is mirrored by the wall and rings at the same time as the first church. The distance l from the first church is constant - we walk along a circle centered at the church. The minimum time difference between the ringing sound and its echo is

$$\Delta t_{\min} = \frac{2h - 2l}{v}\,.$$

This is the time difference if we stand between the two churches. If we move by 180° to the opposite side, the time difference is the maximum

$$\Delta t_{\max} = \frac{2h}{v}\,.$$

Everywhere else on the circle, the time difference Δt satisfies $\Delta t_{\min} < \Delta t < \Delta t_{\max}$. In order to hear ringing with the period $T = 0.5\,\mathrm{s}$, the time difference must be $\Delta t = 0.5\,\mathrm{s}$. If we stand right next to the church ($l = 0\,\mathrm{m}$), then the time difference is greater than $0.5\,\mathrm{s}$. If we move closer to the wall, the time difference decreases until it finally becomes $0.5\,\mathrm{s}$ at the distance

$$l_{\min} = h - \frac{\Delta t v}{2} = 22.5\,\mathrm{m}\,.$$

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Problem O.5 ... soliloquy 5 points

Tomáš was in an empty dormitory without any friends to talk with, so he decided to go jogging and talk to himself. He runs with a constant velocity $10\,\mathrm{km{\cdot}hod^{-1}}$ *along a long wall at a distance* $50\,\mathrm{m}$ *from the wall. He hears his voice not only directly, but also as an echo from the wall. What is the ratio of the frequency of the primary sound to the frequency of the echo?*

Kiko has a friend to talk with.

Since Tomáš runs with constant velocity, the velocities of the transmitter and receiver are the same. Therefore, the frequency of the sound doesn't change and the ratio will be equal to 1.

This result can also be explained when we can replace the wall by a second runner (a mirror image of Tomáš exactly copying his motion) at a distance $100\,\mathrm{m}$ from Tomáš. This situation with two people running in parallel and talking to each other is equivalent to hearing an echo from a wall at the distance $50\,\mathrm{m}$. Relative motion of the air with respect to the runners doesn't change the frequency of the sound. Therefore, the ratio of frequencies is 1.

If you want to see some math, the Doppler shift is

$$f = \frac{c - v_{\mathrm{receiver}}}{c - v_{\mathrm{transmitter}}} f_0\,.$$

The velocities of the receiver v_{detector} and the transmitter $v_{\mathrm{transmitter}}$ are the same, so $f/f_0 = 1$.

The only condition is that Tomáš must be moving slower than sound.

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Problem O.6 ... a bubble in a paperweight 5 points

Jindra was intrigued by a spherical glass paperweight with a radius $r = 5.0\,\mathrm{cm}$. There is an air bubble inside the paperweight at a depth $a = 3.0\,\mathrm{cm}$. Jindra observes the bubble along the ray eye-bubble-centre of the sphere. What is the apparent depth of the bubble that Jindra observes? The refractive index of glass is $n_\mathrm{s} = 1.52$, the refractive index of air is $n_\mathrm{v} = 1.00$.

Instead of preparing for exams, Jindra was observing a bubble.

The object (bubble) is at the distance $a = 3.0\,\mathrm{cm}$ from the spherical glass-air boundary, which refracts the light rays coming from it and creates an image of the bubble at a distance a'. In the small angle approximation, we can derive the relation between these distances

$$\frac{n_\mathrm{g}}{a} + \frac{n_\mathrm{a}}{a'} = \frac{n_\mathrm{g} - n_\mathrm{a}}{r}\,.$$

If a' is negative, the image is inside the paperweight and virtual and if a' is positive, the image is outside the paperweight and real. We get

$$\frac{1}{a'} = \left(\frac{n_\mathrm{g}}{n_\mathrm{a}} - 1\right)\frac{1}{r} - \frac{n_\mathrm{g}}{n_\mathrm{a}}\frac{1}{a}\,,$$

$$a' \doteq -2.48\,\mathrm{cm}\,.$$

Jindra sees the bubble at the depth 2.48 cm.

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Problem T.1 ... drinking water 5 points

Danka had a glass bottle with a volume $V_\mathrm{b} = 0.500\,\ell$ filled with $V_1 = 4.00\,\mathrm{d}\ell$ of water. She put it to her mouth in such a way that air could not flow inside and drank $V_\mathrm{o} = 2.00\,\mathrm{d}\ell$ of water without blowing any air in. Find the decrease in pressure in the bottle. Assume that the process was fast enough that no heat was exchanged between the water, the bottle and the surrounding environment. At the beginning, the pressure of the air in the bottle is the same as the atmospheric pressure $p_\mathrm{a} = 1\,013\mathrm{hPa}$. Assume that air consists of molecules with $f = 5$ degrees of freedom. *Danka's plastic bottle always shrinks.*

Since there is no heat exchange between the air in the bottle and its surroundings, the process is adiabatic, for which $pV^\kappa = \mathrm{const}$ holds. κ is the heat capacity ratio (typically denoted by γ in some countries), which links to the number of degrees of freedom as

$$\kappa = \frac{f+2}{f}\,.$$

Let p_2 denote the air pressure in the bottle after drinking. Then

$$p_\mathrm{a}\left(V_\mathrm{b} - V_1\right)^{\frac{f+2}{f}} = p_2\left(V_\mathrm{b} - V_1 + V_\mathrm{o}\right)^{\frac{f+2}{f}}\,,$$

holds for the air inside the bottle. The decrease in pressure can be found as the difference between the pressure before and after

$$\Delta p = p_\mathrm{a} - p_2\,.$$

By expressing the pressure p_2 from the penultimate equation and plugging it into the last one we obtain

$$\Delta p = p_\mathrm{a}\left(1 - \left(\frac{V_\mathrm{b} - V_1}{V_\mathrm{b} - V_1 + V_\mathrm{o}}\right)^{\frac{f+2}{f}}\right) \doteq 795\,\mathrm{hPa}\,.$$

We see that if the bottle was ideally sealed, the pressure difference would be more than 75 % of atmospheric pressure, which is an over-estimation. Of course, such negative pressure is not achievable by mouth only. However, even a much smaller and actually achievable pressure difference, is enough to shrink the material of a common plastic bottle.

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Problem T.2 ... a little sun 5 points

Determine the temperature of a black body with the same radiated power as the Sun, but with a diameter equal to half of the Sun's diameter $D_\odot = 1\,392\,000\,\mathrm{km}$. The power radiated by the Sun is $P_\odot = 3.826 \cdot 10^{26}\,\mathrm{W}$ and the Stefan-Boltzmann constant is $\sigma = 5.67 \cdot 10^{-8}\,\mathrm{W{\cdot}m^{-2}{\cdot}K^{-4}}$. *Danka was cold.*

The total radiant intensity of a black body $I(T)$ obeys the Stefan-Boltzmann law

$$I(T) = \sigma T^4\,,$$

where T is the temperature of the black body. In order to get the radiated power, we have to multiply the intensity by the surface area of the body

$$S = \pi D^2 \,.$$

We start with the equation $P_\odot = P$, which we can rewrite using the previous identities as

$$P_\odot = \pi D^2 \sigma T^4 \,.$$

Substituting $D = D_\odot/2$ and expressing T, we get

$$T = \sqrt[4]{\frac{4P_\odot}{\pi\sigma D_\odot^2}} \doteq 8\,160\,\mathrm{K} \,.$$

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Problem T.3 ... a simplified model of a balloon — 5 points

Imagine that we have a small deflated balloon with a mass $m = 25\,\mathrm{g}$ made of very thin rubber. When it is inflated, the difference between the pressure inside and the pressure outside is constantly $\Delta p = 10\,\mathrm{kPa}$. We blow $n = 1.0\,\mathrm{mol}$ of air inside it. Assuming normal atmospheric conditions outside the balloon, find the smallest possible temperature inside the balloon such that it would float. Air behaves like an ideal gas and its molar mass is $M = 28.96\,\mathrm{g{\cdot}mol^{-1}}$.

Legolas lives with his head in the clouds.

It follows from the problem statement that inside the balloon, the pressure will be $p = p_\mathrm{a} + \Delta p$. Therefore, we can easily calculate its volume

$$V = \frac{nRT}{p} = \frac{nRT}{p_\mathrm{a} + \Delta p} \,,$$

where we neglect the volume of the balloon itself – this is the volume of the air inside it. We are interested in the amount of air inside

$$n_V = \frac{p_\mathrm{a} V}{RT_\mathrm{a}} = n\frac{p_\mathrm{a} T}{(p_\mathrm{a} + \Delta p)\,T_\mathrm{a}} \,.$$

We must not forget that the air in the balloon also has mass. It remains for us to realise that we get the mass of air by multiplying the amount of air by its molar mass. Then, from equality of the force of gravity and the buoyant force, we get

$$\begin{aligned} F_g &= F_\mathrm{b} \,, \\ g\,(m + Mn) &= gMn_V \,, \\ m + Mn &= Mn\frac{p_\mathrm{a} T}{(p_\mathrm{a} + \Delta p)\,T_\mathrm{a}} \,, \\ T &= T_\mathrm{a}\left(1 + \frac{m}{Mn}\right)\left(1 + \frac{\Delta p}{p_\mathrm{a}}\right) \doteq 600\,\mathrm{K} \,. \end{aligned}$$

We may notice that this temperature depends on the ratio of the balloon's mass to the mass of air inside it. This ratio decreases with the increasing size of the balloon (the mass of the balloon increases quadratically, the amount of air cubically), which is the reason why it's not necessarry to heat the air in large balloons to very high temperatures.

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Problem T.4 ... hot ball 5 points

During a sunny day, we place a ball (hollow, filled with air) with a diameter $d = 20.0\,\mathrm{cm}$ *in front of a house. The initial temperature of the ball is* $t_0 = 20.0\,\mathrm{^\circ C}$*, its heat capacity is* $C = 500\,\mathrm{J\cdot ^\circ C^{-1}}$ *and it absorbs* $\eta = 40\,\%$ *of the solar energy incident on it. The radiant flux from the Sun at the point in front of the house where the ball is lying is* $\Phi = 600\,\mathrm{W\cdot m^{-2}}$*. What is the ratio of the pressure in the ball after half of an hour of lying in the sunshine to the initial pressure in it? Assume that the size of the ball remains constant and it is thermally insulated from its surroundings. Neglect any radiative losses.* *Danka forgot a ball in the sunshine.*

During the time $\tau = 1\,800\,\mathrm{s}$, the ball absorbs energy

$$Q = \frac{\pi d^2}{4}\eta\Phi\tau\,.$$

The temperature of the ball changes by

$$\Delta T = \frac{Q}{C}\,.$$

From the equation of state of an ideal gas for an isochoric process (with constant volume), we get

$$\frac{p}{T} = \mathrm{const}\,,$$

where T is the thermodynamic temperature. Therefore, we get a formula for the ratio of the pressures

$$\frac{p}{p_0} = \frac{T}{T_0} = \frac{T_0 + \Delta T}{T_0} = 1 + \frac{\pi d^2 \eta\Phi\tau}{4CT_0} \doteq 1,09\,.$$

The pressure in the ball increases 1.09 times.

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Problem T.5 ... a breakfast in Dukovany 5 points

Find the weight of ^{238}*Pu (in grams) which we would need to boil* $300\,\mathrm{ml}$ *of water in just three minutes for a breakfast tea while camping. The initial temperature of the water is* $\theta_0 = 15\,\mathrm{^\circ C}$*. We assume that we can utilise* $80\,\%$ *of the energy released by decay. The half-life of* ^{238}*Pu is 87.7 years and it decays with* α *decay with mean released energy* $5.593\,\mathrm{MeV}$*.*
Jarda wanted to go camping, but he didn't want to carry a heavy backpack.

The number of radioactive nuclei satisfies

$$N = N_0 \mathrm{e}^{-\lambda t}\,,$$

where N_0 is the initial number of nuclei, $\lambda = \frac{\ln 2}{T}$ is the decay constant and T is the half-life. Activity (the number of decays per second) is given as the absolute value of the time derivative of the formula above, therefore

$$A = N_0 \lambda \mathrm{e}^{-\lambda t} = \frac{N_0 \ln 2}{T} \mathrm{e}^{-\lambda t} \,,$$

where the absolute value was used in order to obtain the "count" – otherwise the number would be negative, because it is a decrease. If we multiply the activity by the mean energy released per decay and the efficiency $\eta = 0.8$, we will obtain the thermal power which can heat the water. In $t = 3\,\mathrm{min} = 180\,\mathrm{s}$ the water receives the heat $Q = mc\Delta T = mc\left(\theta_\mathrm{b} - \theta_0\right)$, where $m = 0.3\,\mathrm{kg}$, $c = 4\,200\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ is the specific heat capacity of water and $\theta_\mathrm{b} = 100\,\mathrm{^\circ C}$ is the boiling point of water. Since $t \ll T$, we assume $\mathrm{e}^{-\lambda t} \approx 1$ and therefore approximate the activity to be constant (otherwise we would have to integrate the activity in time – the integral is not hard, but we like to keep it simple). From the information given we get the equation

$$E \eta N_0 \frac{\ln 2}{T} = \frac{mc\left(\theta_\mathrm{b} - \theta_0\right)}{t} \,,$$

from which we express the number of nuclei

$$N_0 = \frac{mc\left(\theta_\mathrm{b} - \theta_0\right) T}{\eta E t \ln 2} \doteq 3.312 \cdot 10^{24} \,.$$

The molar mass of $^{238}\mathrm{Pu}$ is $M = 238.05\,\mathrm{g}$. Therefore we need

$$m = \frac{N_0 M}{N_\mathrm{A}} = 1\,309\,\mathrm{g} \,.$$

We see that to generate almost $600\,\mathrm{W}$ we need only $1.3\,\mathrm{kg}$ of the material, which is not that much. It is widely used in so-called radioisotope thermoelectric generators (RTGs). $^{238}\mathrm{Pu}$ is used (or has been used), for example, in Curiosity rover, space probes Cassini, New Horizons or lunar modules from the Apollo programme. RTGs in general need isotopes which emit particles of high enough energy, have an optimal half-life (too short would be less useful due to short lifespan, too high would emit too little radiation and therefore the amount needed would be too high). Furthermore, the radiation has to decelerate rapidly – otherwise the energy will escape the device and the generator will be both inefficient and dangerous for the crew or electric systems. The α-particle is big and heavy, therefore it decelerates very quickly. Therefore, the decay heat remains in the material and a pellet of $^{238}\mathrm{Pu}$ can heat itself to several hundred °C.

In the end, we should note that each radioactive emitter shall be dealt with according to the principles of radiation protection. This, among others, posits that radioisotopes should only be used if they provide sufficient value and there is no safer alternative. Our problem was purely educational – a small camping gas stove is a much better alternative.

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Problem T.6 ... oscillating piston — 5 points

Suppose that we have an insulated closed cylinder with a length $2l = 2.0\,\mathrm{m}$ and a constant cross-section $S = 100\,\mathrm{cm}^2$ lying on the ground. We add a small partition (wall) with a mass $m = 2.0\,\mathrm{kg}$

inside in such a way that there are $N = 10^{23}$ molecules of an ideal gas with a heat capacity ratio $\kappa = 5/3$ on each side of the partition. The pressure settles on an equilibrium value $p_0 = 100\,\mathrm{kPa}$. Find the period of small oscillations of the partition.

Lego likes small . . . oscillations.

The cylinder is thermally insulated, so the thermodynamic processes inside it will be adiabatic. For such processes, $pV^\kappa = \mathrm{const}$ applies. Since there is the same amount of gas on both sides of the partition, its mechanical equilibrium will be in the middle of the cylinder. Then we can calculate the pressure difference between the two separated chambers when the partition moves by a very small distance x. From the equation $p_0 V_0^\kappa = p_x V_x^\kappa$, we express

$$p_x = p_0 \frac{V_0^\kappa}{V_x^\kappa} = p_0 \frac{S^\kappa l^\kappa}{S^\kappa (l+x)^\kappa} = p_0 \left(1 + \frac{x}{l}\right)^{-\kappa} \approx p_0 \left(1 - \kappa \frac{x}{l}\right) .$$

We must not forget that in the other chamber, the pressure will also change, the partition will just move in the opposite direction. Therefore, we can obtain the pressure in it almost in the same way, we only have to change $x \to -x$. Then we can calculate the resultant force, which acts on the partition when it is displaced from the mechanical equilibrium by x, as

$$F_x = S\Delta p_x = S\left(p_0\left(1 - \kappa\frac{x}{l}\right) - p_0\left(1 + \kappa\frac{x}{l}\right)\right) = -\frac{2\kappa S p_0}{l} x .$$

Let us also check the direction. In the chamber towards which the partition moves, the pressure increases (so it pushes harder on the partition), and in contrast, in the other chamber the pressure decreases. Therefore, the force acts against the direction of the partition's displacement and small oscillations will occur.

It only remains to realise that for the stiffness k of a linear harmonic oscillator, $F = -kx$ applies. Therefore we can identify $k = 2\kappa S p_0 / l$. By substituting into the formula for the period of small oscillations, we get

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ml}{2\kappa S\ p_0}} \doteq 0.15\,\mathrm{s} .$$

We may notice that we have not used the amount of the gas in the cylinder nor its temperature.

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Problem T.7 . . . a mining poison 5 points

Imagine a cylindrical tank with a height $H = 100\,\mathrm{m}$ and base radius $r = 20\,\mathrm{m}$. It is closed and filled with air (78 % of the volume is nitrogen, 21 % oxygen and 1 % argon) with a temperature $T = 27\,^\circ\mathrm{C}$ and pressure $p = 1\,013\,\mathrm{hPa}$. We burn some amount of pure carbon inside the tank, which transforms exactly one third of the oxygen molecules into carbon dioxide. Find the ratio of the molar concentration of oxygen to the molar concentration of carbon dioxide at the bottom of the tank when the temperature settles at its original value and the system reaches an equilibrium.

Dodo sleeps on the upper bunk.

The first important thing is to realise that the gases in the box will be distributed independently of each other.[3] Each of these gases is distributed in the box in equilibrium following the Boltzmann distribution.

$$c(h) = c_0 \mathrm{e}^{-\frac{E_\mathrm{p}}{k_\mathrm{B}T}} = c_0 \mathrm{e}^{-\frac{mgh}{k_\mathrm{B}T}} \,,$$

where m is the mass of the gas molecule, c_0 is the molar concentration at the bottom of the box and $c(h)$ is the molar concentration at height h. This relation can be derived, for example, for the pressure evolution with height from the equation of hydrostatic equilibrium

$$\mathrm{d}p = -\rho g \,\mathrm{d}h$$

and from the equation of state of an ideal gas

$$p = \frac{\rho}{M} RT \,.$$

The value of the molar concentration at the bottom of the box is determined from the condition for the total substance amount of the gases

$$n = \int_0^H S c(h) \,\mathrm{d}h = S c_0 \frac{k_\mathrm{B}T}{mg} \left(1 - \mathrm{e}^{-\frac{mgH}{k_\mathrm{B}T}}\right) \,,$$

$$c_0 = \frac{nmg}{Sk_\mathrm{B}T} \left(1 - \mathrm{e}^{-\frac{mgH}{k_\mathrm{B}T}}\right)^{-1} \approx \frac{nmg}{Sk_\mathrm{B}T} \left(\frac{mgH}{k_\mathrm{B}T} - \frac{1}{2}\left(\frac{mgH}{k_\mathrm{B}T}\right)^2\right)^{-1} = \frac{n}{SH}\left(1 - \frac{mgH}{2k_\mathrm{B}T}\right)^{-1} \,,$$

where we used the approximation $\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2$. This formula is applicable to each of the gases separately. Let us mark the oxygen with the index c^o, carbon dioxide with c^d and carbon with c^c. We get the required quotient as

$$\frac{c_0^\mathrm{o}}{c_0^\mathrm{d}} = \frac{\frac{n^\mathrm{o}}{SH}\left(1 - \frac{m^\mathrm{o}gH}{2k_\mathrm{B}T}\right)^{-1}}{\frac{n^\mathrm{d}}{SH}\left(1 - \frac{m^\mathrm{d}gH}{2k_\mathrm{B}T}\right)^{-1}} = \frac{n^\mathrm{o}}{n^\mathrm{d}} \frac{1 - \frac{m^\mathrm{d}gH}{2k_\mathrm{B}T}}{1 - \frac{m^\mathrm{o}gH}{2k_\mathrm{B}T}} \approx \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{m^\mathrm{d}gH}{2k_\mathrm{B}T}\right)\left(1 + \frac{m^\mathrm{o}gH}{2k_\mathrm{B}T}\right) \approx$$

$$\approx \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{\left(m^\mathrm{d} - m^\mathrm{o}\right) gH}{2k_\mathrm{B}T}\right) = \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{m^\mathrm{c}gH}{2k_\mathrm{B}T}\right) = \frac{n^\mathrm{o}}{n^\mathrm{d}} \left(1 - \frac{M^\mathrm{c}gH}{2RT}\right) \,.$$

After substituting in the molar mass of carbon $M^\mathrm{c} = 12\,\mathrm{g{\cdot}mol^{-1}}$ we get

$$\frac{c_0^\mathrm{o}}{c_0^\mathrm{d}} \doteq 2\,(1 - 0.002\,36) \doteq 1.995\,3 \,.$$

The result may seem surprising - You probably saw experiments where carbon dioxide was poured from one box to another box. However, if we kept this carbon dioxide alone for a sufficiently long time, diffusion would do its work and it would distribute the carbon dioxide to higher levels due to thermal motion.

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[3] We consider them to be ideal gases.

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